A Finite-Size Square-Law Visibility Suppression in the Bose–Marletto–Vedral Experiment from Quantum-Geometric Correspondence

Quantum-Geometric Correspondence

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Abstract

The Bose–Marletto–Vedral (BMV) / QGEM experiment aims to witness the quantum nature of gravity through the entanglement it generates between two masses in spatial superposition. Quantum-Geometric Correspondence (QGC) predicts that this witness is suppressed by gravitational self-decoherence of the source masses. For solid spheres of radius R in the experimentally relevant regime $\Delta \gg 2R$ (d the inter-particle separation, Δ the superposition arm), the Diósi–Penrose branch-pair self-energy *saturates* at $\approx 1.2\, Gm^2/R$, fixed by particle size rather than arm separation, and the visibility follows a parameter-free *square law* $V(\tau_{\text{BMV}}) = \exp\left[-\mathcal{O}(1)(d/\Delta)^2(d/R)\right]$ with a residual mass trend $\propto m^{-1/3}$ — opposite in sign to the explicit m^2 scaling of continuous-spontaneous-localisation and Diósi–Penrose collapse models. The suppression holds for the full one-parameter family of arm orientations, whose angular dependence is a Legendre $P_2(\cos\theta)$: at the magic angle $\cos^2\theta_m = 1/3$ the leading dipole contribution to the witness phase vanishes identically and the visibility follows a *quartic* $\exp[-(54\pi/35)(d/\Delta)^4(d/R)]$. At the QGEM design point all geometries predict near-total decoherence; the experimental discriminator is the parametric structure — the scaling exponent (square versus the orientation-blind constancy of collapse models), the magic-angle phase null, and the $m^{-1/3}$ trend — which distinguishes QGC from all extant collapse models in a single multi-parameter experiment. The magic-angle structure is robust to experimental imperfection: a 1° arm-orientation spread protects the quartic law across the feasible window, percent-level arm-length jitter is benign, and just off the magic angle ($\varepsilon = \cos^2\theta - 1/3 > 0$) a one-sided anti-resonance crashes the visibility at $\Delta/d = \sqrt{27\varepsilon/7}$ — an orientation-locked signature no collapse model reproduces. Observation of the standard BMV signal at QGEM-class parameters would falsify QGC's constrained-IF picture.

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1 Introduction

The Bose–Marletto–Vedral (BMV) proposal [1, 2] makes the quantumness of gravity a tabletop question: two massive particles, each in spatial superposition and near enough that their mutual gravitational interaction entangles them, witness an entanglement that no classical field could mediate [2, 4, 15]. The QGEM design [3] realises this with nanoparticles in Stern–Gerlach interferometers at masses $\sim 10^{-14}$–10^{-13} kg and arms $\sim 100\ \mu$m, extending matter-wave and optomechanical superposition experiments [13, 12] into the regime where gravity entangles. Quantum-Geometric Correspondence (QGC) predicts that the same setup self-decoheres: gravitational self-decoherence of each particle suppresses the witnessed visibility, by a *square law* $V(\tau_{\rm BMV}) = \exp[-\mathcal{O}(1)(d/\Delta)^2(d/R)]$ for a solid sphere of radius R in the regime $\Delta \gg 2R$, with a residual mass trend $\propto m^{-1/3}$ and a characteristic orientation dependence. This paper derives that prediction and shows how it discriminates QGC from both the standard quantum-gravity expectation and from collapse models.

A standard quantum-gravity treatment of the BMV setup predicts that the entanglement reaches the witness threshold after a time

$$\tau_{\rm BMV} \;\sim\; \frac{\hbar\, d^3}{G m^2 \Delta^2}\,, \tag{1}$$

where d is the inter-particle separation and Δ the superposition arm. Throughout this time the gravitational field is treated as a passive mediator of phase: it neither decoheres the source masses nor itself fluctuates appreciably.

In QGC's constrained Feynman–Vernon influence functional [19], the linearised Wheeler–DeWitt constraint forces the field into a dressed-coherent state for each matter configuration, giving a non-vanishing self-decoherence rate at first order in G, set by the Diósi–Penrose branch-pair self-energy [5, 7]. For a *finite-size* sphere of radius R this self-energy *saturates* once the branches no longer overlap ($\Delta \geq 2R$):

$$\Gamma_{\rm self}(\Delta) \;=\; \frac{E_G^{\rm self}(\Delta)}{\hbar} \;=\; \frac{G m^2}{\hbar}\Big[\frac{6}{5R} - \frac{1}{\Delta}\Big] \;\xrightarrow{\Delta \gg 2R}\; \frac{6}{5}\,\frac{G m^2}{\hbar R} \quad \text{per particle}\,, \tag{2}$$

fixed by the particle *size* R rather than the arm separation Δ. The point-mass form $Gm^2/(\hbar\Delta)$ is recovered only at the boundary $\Delta \to 2R^+$; it never applies to a solid particle at QGEM separations ($\Delta \gg 2R$). The canonical core paper [19] derives this per-particle rate from the linearised Hamiltonian constraint — a G^1 identification that remains a conjecture at the operator level, on which every prediction here is conditional (Section 7); its headline single-mass scale $E_G = GM^2/d$ is the point-particle limit of $E_G^{\rm self}$.

Combining these two ingredients — the standard BMV phase formula and the QGC per-particle decoherence rate — fixes the visibility of the BMV witness. At the witness criterion ($\phi_{\rm BMV} = \pi/2$, when the entanglement is half-formed), the off-diagonal element of the two-particle reduced density matrix is suppressed by a visibility factor that, in the deep-saturated regime $\Delta \gg 2R$, follows a square law

$$V(\tau_{\rm BMV}) \;=\; \exp\Big[-\mathcal{O}(1)\,(d/\Delta)^2\,(d/R)\Big]\,. \tag{3}$$

The square law arises algebraically: substituting the saturated rate $\Gamma_{\rm self} \to (6/5)Gm^2/(\hbar R)$ into

$V = \exp(-2\Gamma_{\rm self}\tau_{\rm BMV})$ with $\tau_{\rm BMV} \propto d^3/\Delta^2$ yields the exponent $\propto d^3/(R\Delta^2)$. The Gm^2 in $\Gamma_{\rm self}$ and $\tau_{\rm BMV}$ cancels, but the residual R in the saturated self-energy leaves a weak mass dependence $\propto m^{-1/3}$ (at fixed density). The numerical prefactor depends only on the arm geometry: $3\pi/5$ for parallel-axis arms and $6\pi/5$ for perpendicular (QGEM "diamond") arms. Had the particle been a point mass, the rate $Gm^2/(\hbar\Delta)$ would have given the steeper cube law $(d/\Delta)^3$; that limit survives only at the unphysical edge $\Delta \to 2R^+$, never at QGEM separations.

Three features make Eq. (3) an unusually clean experimental discriminator. First, the residual mass trend is weak and qualitatively distinct: the suppression scales as $m^{-1/3}$ (heavier, hence larger, spheres decohere *less*), opposite in sign to the explicit m^2 scaling of the Diósi–Penrose rate [5, 7, 6], continuous spontaneous localisation [8], and Penrose's reduction-time hypothesis [7]. Second, the prediction is parameter-free within QGC's constrained influence functional – there is no analogue of the CSL collapse rate λ to fit. Third, the angular dependence of the leading BMV phase is a Legendre $P_2(\cos\theta)$ in the arm-to-axis angle, with the parallel-axis ($\theta = 0$) and perpendicular ($\theta = \pi/2$) geometries differing by a factor of two in the leading exponent and by the sign of the next-to-leading correction. Between these two cardinal orientations sits the magic angle $\cos^2\theta_m = 1/3$ where the dipole term vanishes identically and a higher-multipole quartic law takes over. An experiment that can rotate its arms relative to the inter-particle axis therefore sweeps three qualitatively distinct QGC signatures with the same hardware: square suppression with prefactor $3\pi/5$ at $\theta = 0$, square suppression with prefactor $6\pi/5$ at $\theta = \pi/2$, and quartic suppression with prefactor $54\pi/35$ at $\theta = \theta_m$.

Section 2 sets up the two-mass dynamics in QGC's constrained-IF picture and introduces the finite-size saturated self-energy. Section 3 derives the leading-order square law for both cardinal arm geometries. Section 4 computes the exact-phase corrections, identifies a sign-flip in the sub-leading term between them, and notes that the perpendicular formula has no singularity at $\Delta \to d$ where the parallel formula breaks down. Section 5 extends the analysis to arbitrary arm angle θ, identifies the magic angle, and derives both the leading-order quartic visibility and its exact-phase form. Section 6 gives explicit numerical predictions for QGEM's proposed parameters in all three geometries and compares with competing theories. Section 7 discusses limitations, falsification routes, and the place of this prediction among other near-term tests.

2 The Two-Mass Dynamics in QGC's Constrained Influence Functional

The constrained influence functional dresses each matter configuration with its own coherent gravitational state; tracing out the field leaves a Newtonian BMV phase and a finite-size self-decoherence factor whose energy scale saturates at the particle size.

2.1 State and field

Two equal masses m, labelled A and B, are each prepared in a balanced spatial superposition,

$$|\psi_A\rangle \;=\; \frac{1}{\sqrt{2}}\big(|L\rangle_A + |R\rangle_A\big)\,, \qquad |\psi_B\rangle \;=\; \frac{1}{\sqrt{2}}\big(|L\rangle_B + |R\rangle_B\big)\,, \tag{4}$$

with arm separation Δ and centre-to-centre separation d between the two particles. Two arm geometries are of experimental interest:

- *Parallel-axis* – the two arms of each particle lie along the inter-particle axis; the four branch distances are $r_{LL} = r_{RR} = d$ and $r_{LR} = d - \Delta$, $r_{RL} = d + \Delta$ (so the minimum branch distance is $d - \Delta$, requiring $\Delta < d$).
- *Perpendicular-arm* (QGEM "diamond") – both arms of each particle lie in a plane transverse to the inter-particle axis, displaced by $\pm\Delta/2$ from the central line; the branch distances are $r_{LL} = r_{RR} = d$ and $r_{LR} = r_{RL} = \sqrt{d^2 + \Delta^2}$ (no constraint between Δ and d).

The combined matter state has four computational-basis amplitudes,

$$|\Psi^{\text{matter}}\rangle \;=\; \frac{1}{2} \sum_{i,j\in\{L,R\}} |i\rangle_A\, |j\rangle_B \;, \tag{5}$$

each of which sources its own gravitational configuration. In the constrained Feynman–Vernon picture [19], integrating out the linearised gravitational degrees of freedom subject to the Wheeler–DeWitt constraint replaces this state with a dressed total state

$$|\Psi(t)\rangle \;=\; \frac{1}{2} \sum_{i,j} |i\rangle_A\, |j\rangle_B \; e^{-i\phi_{ij}(t)} \; |\alpha_{ij}\rangle \;, \tag{6}$$

where $|\alpha_{ij}\rangle$ is the constraint-enforced dressed coherent state of the gravitational field for matter configuration (i, j) and the phase $\phi_{ij}(t)$ is the Newtonian gravitational interaction phase for that configuration:

$$\phi_{ij}(t) \;=\; -\,\frac{E_{ij}\, t}{\hbar} \,, \qquad E_{ij} \;=\; -\,\frac{Gm^2}{r_{ij}} \,. \tag{7}$$

2.2 Reduced matter density matrix

Tracing over the gravitational field,

$$\rho^{\text{matter}}_{ij,kl}(t) \;=\; \frac{1}{4}\, \langle \alpha_{kl} |\, \alpha_{ij}\rangle \; e^{-i(\phi_{ij} - \phi_{kl})\, t/\hbar} \,. \tag{8}$$

Two distinct objects appear:

- the *phase* $\phi_{ij} - \phi_{kl}$ is the standard BMV entanglement phase, generating coherent two-particle correlations as t grows;
- the *dressed-state overlap* $\langle \alpha_{kl} |\, \alpha_{ij}\rangle$ is the QGC-specific decoherence factor. In the conventional unconstrained Feynman–Vernon analysis this overlap equals unity (the gravitational field is pure noise that adds no off-diagonal damping at first order in G). In the constrained analysis it is a non-trivial Gaussian.

2.3 The QGC decoherence factor

Two coherent states satisfy $|\,\langle \alpha_a |\, \alpha_b\rangle| = \exp(-\|\delta\alpha_{ab}\|^2/2)$. In the constrained-IF picture the squared overlap exponent for two configurations differing by displacement of a single mass m over a distance Δ saturates [19] at

$$\|\delta\alpha\|^2_{\text{sat}} \;=\; \xi\, \ln(d/\ell_P)\,, \qquad \xi \;=\; \frac{Gm^2}{\hbar c}\,, \tag{9}$$

with ℓ_P the Planck length. Equation (9) is the *static* overlap between the dressed fields of two *fixed* configurations: the field-dressing transient equilibrates after the sound-crossing time d/c and contributes no further coherence loss. The decoherence relevant to the witness is a distinct object — the loss of coherence between the two branches as they persist in superposition — which the constraint enforces as a continued growth of the branch-pair overlap exponent, linear in time at the per-particle rate $\Gamma_{\rm self}(\Delta) = E_G^{\rm self}(\Delta)/\hbar$, set by the gravitational self-energy of one mass split into two spatially separated branches.

The energy scale that drives this growth is the Diósi–Penrose branch-pair self-energy [5, 7]. For a *point* mass it is the mutual energy Gm^2/Δ, which diverges as $\Delta \to 0$. A real BMV particle is a solid sphere of radius R, and its branch-pair self-energy saturates once the two branches no longer overlap.

Proposition 2.1 (Finite-size self-energy saturation)**.** *For a uniform sphere of radius $R = (3m/4\pi\rho)^{1/3}$ split into two branches of separation Δ, the branch-pair self-energy is C^1-continuous at $\Delta = 2R$ (value $\frac{7}{10}Gm^2/R$) and saturates at $\frac{6}{5}Gm^2/R$ for $\Delta \geq 2R$, fixed by the particle size rather than the arm separation.*

The exact uniform-sphere self-energy is

$$E_G^{\rm self}(\Delta) \;=\; \begin{cases} \dfrac{Gm^2}{R}\left[\,2t^2 - \frac{3}{2}t^3 + \frac{1}{5}t^5\,\right], & \Delta < 2R\,, \quad t \equiv \Delta/2R\,, \\[2ex] \dfrac{6}{5}\dfrac{Gm^2}{R} \;-\; \dfrac{Gm^2}{\Delta}\,, & \Delta \geq 2R\,, \end{cases} \tag{10}$$

which grows as $\sim Gm^2\Delta^2/(2R^3)$ at small Δ and saturates at

$$E_G^{\rm self}(\Delta \to \infty) \;\to\; \frac{6}{5}\,\frac{Gm^2}{R} \;=\; 1.2\,\frac{Gm^2}{R}\,. \tag{11}$$

The per-particle self-decoherence rate is

$$\Gamma_{\rm self}(\Delta) \;=\; \frac{E_G^{\rm self}(\Delta)}{\hbar} \;=\; \frac{Gm^2}{\hbar}\Big[\,\frac{6}{5R} - \frac{1}{\Delta}\,\Big] \qquad (\Delta \geq 2R)\,. \tag{12}$$

We adopt the natural coefficient $C = 1$ (Markovian dephasing); the canonical window $C \in [2/\pi, 1]$, whose floor is set by the Margolus–Levitin bound, simply rescales every exponent prefactor by C and leaves the geometric scaling untouched. The canonical core paper [19] derives the per-particle linear-growth rate from the linearised Hamiltonian constraint; that G^1 identification is the framework's central conjecture (Section 7), and its headline single-mass result $E_G = GM^2/d$ is the point-particle limit of Eq. (10), recovered when the branch separation is taken much larger than the source but the source is itself treated as point-like.

QGEM is deep in the saturated regime. For QGEM-class silica spheres ($m \sim 10^{-14}$–10^{-13} kg, $\rho = 2200$ kg/m^3, so $R \sim 1$–2 μm) with $d \sim 120$–200 μm and $\Delta \sim 0.5d$–$2d$, one has $\Delta/2R \approx 14$–$195 \gg 1$: the point-mass form Gm^2/Δ never applies, and the relevant energy is the saturated value $\approx 1.2\,Gm^2/R$, fixed by the particle *size* R rather than by the arm separation Δ. Throughout this work the experimentally relevant regime is therefore both temporally saturated ($\tau_{\rm BMV} \gg d/c \sim 10^{-12}$ s) and spatially saturated ($\Delta \gg 2R$).

2.4 The competition of two clocks

The BMV witness requires the off-diagonal phase $\phi_{ij} - \phi_{kl}$ to grow large enough to register. QGC sets a second clock against it: the dressed-state overlap $|\langle\alpha_{kl}|\,\alpha_{ij}\rangle|$ decays exponentially over the same wait, at rate $2\Gamma_{\text{self}}(\Delta)$ for the channels in which both particles change configuration (e.g., $LL \leftrightarrow RR$). The visibility of the off-diagonal element at the time the witness criterion is reached, in the saturated regime $\Delta \geq 2R$ that QGEM occupies, is the object of the rest of the paper.

3 The Square Law: Leading-Order Derivation for Both Geometries

Substituting the saturated rate into the BMV witness time turns the point-mass cube law into a square law in d/Δ for both cardinal arm orientations.

3.1 Parallel-axis arms

For arms aligned with the inter-particle axis, the Newtonian interaction energies of the four configurations are $E_{LL} = E_{RR} = -Gm^2/d$ and $E_{LR} = -Gm^2/(d-\Delta)$, $E_{RL} = -Gm^2/(d+\Delta)$. The off-diagonal phase between LL and RR is $\phi_{LL} - \phi_{RR} = 0$ (both diagonal channels share the same energy); the relevant phase for the witness is the one between the two "crossed" channels and the diagonal:

$$\Delta\phi^{\text{lead}}_{\text{par}}(t) \;=\; \frac{Gm^2 t}{\hbar}\left[\frac{2}{d} - \frac{1}{d-\Delta} - \frac{1}{d+\Delta}\right] \;=\; -\,\frac{Gm^2 t}{\hbar}\,\frac{2\Delta^2}{d(d^2-\Delta^2)}\,. \tag{13}$$

The leading-order expansion in Δ/d gives the standard BMV formula:

$$\Delta\phi^{\text{lead}}_{\text{par}}(t) \;\simeq\; \frac{2\,Gm^2\,\Delta^2\,t}{\hbar\,d^3}\,. \tag{14}$$

The BMV witness criterion is reached when this phase equals $\pi/2$:

$$\tau^{\text{par,lead}}_{\text{BMV}} \;=\; \frac{\pi\hbar\,d^3}{4\,Gm^2\,\Delta^2}\,. \tag{15}$$

The QGC visibility at this time, using the saturated per-particle self-decoherence rate $\Gamma_{\text{self}}(\Delta) = (Gm^2/\hbar)[6/(5R) - 1/\Delta]$ from Eq. (12) (valid for $\Delta \geq 2R$, the QGEM regime), is the dressed-state-overlap modulus

$$V^{\text{par}}_{\text{lead}}(\tau^{\text{par,lead}}_{\text{BMV}}) \;=\; \exp\!\Big(-\,2\Gamma_{\text{self}}(\Delta)\,\tau^{\text{par,lead}}_{\text{BMV}}\Big) \;=\; \exp\!\Big[-\,\frac{\pi}{10}\,\frac{d^3(6\Delta-5R)}{\Delta^3 R}\Big]\,. \tag{16}$$

The factor of two in $2\Gamma_{\text{self}}$ counts the two particles that each contribute coherence loss to the off-diagonal element. The Gm^2 in the rate cancels against the $1/(Gm^2)$ in τ_{BMV}, but the residual R in the saturated self-energy carries a weak mass dependence (Eq. (17) below). In the deep-saturated regime $\Delta \gg 2R$ relevant to QGEM, the $-5R$ in Eq. (16) is negligible against 6Δ

and the exponent reduces to a *square* law in d/Δ:

$$-\ln V_{\text{lead}}^{\text{par}} \xrightarrow{\Delta\gg 2R} \frac{3\pi}{5}\,\frac{d^3}{R\,\Delta^2} \;=\; \frac{3\pi}{5}\,(d/\Delta)^2\,\frac{d}{R}\,. \tag{17}$$

3.2 Perpendicular-arm (QGEM "diamond") geometry

For arms transverse to the inter-particle axis, the four configurations have $E_{LL} = E_{RR} = -Gm^2/d$ and $E_{LR} = E_{RL} = -Gm^2/\sqrt{d^2+\Delta^2}$. The witness phase difference is

$$\Delta\phi_{\text{perp}}(t) \;=\; \frac{2\,Gm^2 t}{\hbar}\left[\frac{1}{d} - \frac{1}{\sqrt{d^2+\Delta^2}}\right]. \tag{18}$$

At leading order in Δ/d,

$$\Delta\phi_{\text{perp}}^{\text{lead}}(t) \;\simeq\; \frac{Gm^2\,\Delta^2\,t}{\hbar\,d^3}\,, \tag{19}$$

which is exactly half the parallel-axis leading-order formula (14). The phase therefore takes twice as long to reach the $\pi/2$ witness criterion:

$$\tau_{\text{BMV}}^{\text{perp,lead}} \;=\; \frac{\pi\hbar\,d^3}{2\,Gm^2\,\Delta^2} \;=\; 2\,\tau_{\text{BMV}}^{\text{par,lead}}\,. \tag{20}$$

Substituting into the visibility formula gives, with the same saturated self-energy,

$$V_{\text{lead}}^{\text{perp}}(\tau_{\text{BMV}}^{\text{perp,lead}}) \;=\; \exp\!\Big[-\frac{\pi}{5}\,\frac{d^3(6\Delta-5R)}{\Delta^3 R}\Big] \xrightarrow{\Delta\gg 2R} \exp\!\Big[-\frac{6\pi}{5}\,\frac{d^3}{R\,\Delta^2}\Big]\,, \tag{21}$$

exactly twice the suppression exponent of the parallel-axis case at the same Δ/d and R.

3.3 The square law

Proposition 3.1 (Deep-saturated square law)**.** *In the deep-saturated regime $\Delta \gg 2R$, both cardinal leading-order visibilities take the form*

$$V_{\text{lead}}^{(g)} \;=\; \exp\!\Big[-c_g\,(d/\Delta)^2\,(d/R)\Big]\,, \qquad c_{\text{par}} = \frac{3\pi}{5}\,, \qquad c_{\text{perp}} = \frac{6\pi}{5}\,, \tag{22}$$

with the prefactor c_g fixed by the arm geometry g alone.

The scaling is a *square* law in d/Δ with an explicit factor d/R, not the cube law $(d/\Delta)^3$ of the point-mass rate Gm^2/Δ. The change of power traces to the self-energy saturation: the energy driving self-decoherence is $\approx (6/5)Gm^2/R$, set by the particle *size* R, not Gm^2/Δ, set by the arm separation. The cube law survives only as the $\Delta \to 2R^+$ edge behaviour (Section 4), where the two branches just cease to overlap; it never applies to a solid sphere at QGEM separations. Three features follow:

1. *Residual mass trend $m^{-1/3}$.* The bulk of the mass dependence cancels ($\Gamma_{\text{self}} \propto m^2$ against $\tau_{\text{BMV}} \propto 1/m^2$), but the saturated self-energy retains the factor $1/R$, and at fixed density $R = (3m/4\pi\rho)^{1/3} \propto m^{1/3}$. Hence at fixed geometry (d, Δ, ρ)

$$-\ln V \;\propto\; \frac{1}{R} \;\propto\; m^{-1/3}\,, \tag{23}$$

a slow, definite trend: heavier spheres are larger, saturate at a lower self-energy, and so decohere *less*. This is weaker than the strict mass-independence of the old point-mass cube law, but it remains qualitatively distinct from the Diósi–Penrose collapse rate [5, 7] ($\propto m^2$), continuous spontaneous localisation [8] (squared local density), and Penrose's reduction-time formula ($\propto m^2$). A mass scan at fixed (d, Δ, ρ) is therefore itself a discriminator.

2. *Parameter freedom.* The prefactor c_g is fixed by the geometry alone; QGC has no analogue of the CSL collapse rate λ to fit. Up to the small ambiguity of choosing the witness criterion at $\phi = \pi/2$ versus $\phi = \pi$ (which doubles c_g), the prediction is rigid.

3. *Geometry sensitivity.* Choosing parallel versus perpendicular arms doubles the exponent at the same Δ/d and R. An experiment that can run with both arm orientations therefore provides an internal cross-check of the mechanism.

Section 4 treats the corrections at $\Delta \sim d$ (the QGEM operating regime), where the leading-order phase is inadequate and the unsaturated $\Delta \to 2R^+$ edge restores the cube power with an enhanced coefficient.

4 Exact-Phase Corrections and the Sign-Flip Between Geometries

The leading-order square law of Section 3 assumes $\Delta \ll d$ in the BMV *phase*, whereas QGEM operates at $\Delta \sim d$. Using the exact branch energies of Eqs. (13) and (18) corrects only the witness time $\tau_{\rm BMV}$; the decoherence rate stays the saturated $\Gamma_{\rm self}(\Delta) = (Gm^2/\hbar)[6/(5R) - 1/\Delta]$ of Eq. (12) throughout ($\Delta \gg 2R$). The two geometries develop sub-leading (Δ/d) phase corrections of *opposite sign*.

4.1 Parallel-axis exact-phase visibility

Setting the exact phase (13) equal to $\pi/2$ gives

$$\tau_{\rm BMV}^{\rm par,exact} = \frac{\pi\hbar\, d\,(d^2 - \Delta^2)}{4\,Gm^2\,\Delta^2} = \tau_{\rm BMV}^{\rm par,lead}\left(1 - (\Delta/d)^2\right), \tag{24}$$

The exact phase reaches the witness threshold *faster* than the leading-order formula, since the singularity at $\Delta \to d$ enhances phase accumulation; QGC has less time to decohere the superposition, and the prediction is more permissive. Multiplying $2\Gamma_{\rm self}(\Delta)$ into Eq. (24) (the Gm^2 cancels) gives, exactly,

$$-\ln V_{\rm exact}^{\rm par} = \frac{\pi}{2}\,\frac{d^3}{\Delta^2}\left(1 - (\Delta/d)^2\right)\left[\frac{6}{5R} - \frac{1}{\Delta}\right], \tag{25}$$

which in the deep-saturated regime $\Delta \gg 2R$ reduces to a square law with a $1 - (\Delta/d)^2$ phase correction,

$$-\ln V_{\rm exact}^{\rm par} \xrightarrow{\Delta \gg 2R} \frac{3\pi}{5}\,\frac{d^3}{R\,\Delta^2}\left(1 - (\Delta/d)^2\right). \tag{26}$$

The ratio to the leading-order square law (16) is

$$\frac{V^{\rm par}_{\rm exact}}{V^{\rm par}_{\rm lead}} \;=\; \exp\!\Big[+\frac{3\pi}{5}\,\frac{d}{R}\Big]\,, \tag{27}$$

so the exact phase makes the parallel-axis prediction more permissive by a factor that grows with d/R. Equation (25) reduces to the leading square law in the wide-arm limit $\Delta/d \to 0$ and to $V \to 1$ as $\Delta \to d^-$, where the parallel-axis arms touch the partner particle and the linearised Newtonian phase formula breaks down. The expression is well-defined only for $\Delta < d$ (and, as always, $\Delta \geq 2R$).

4.2 Perpendicular-arm exact-phase visibility

Setting Eq. (18) equal to $\pi/2$ and writing $u \equiv \Delta/d$, one finds

$$\tau^{\rm perp,exact}_{\rm BMV} \;=\; \frac{\pi\hbar\, d}{4\,Gm^2}\,\frac{\sqrt{1+u^2}}{\sqrt{1+u^2}-1}\,. \tag{28}$$

After rationalising and substituting $2\Gamma_{\rm self}(\Delta)$ (see Appendix A), the exact perpendicular exponent is

$$-\ln V^{\rm perp}_{\rm exact}(u) \;=\; \frac{\pi}{2}\,d\,\frac{1+u^2+\sqrt{1+u^2}}{u^2}\,\Big[\frac{6}{5R}-\frac{1}{\Delta}\Big]\,, \tag{29}$$

which in the deep-saturated regime $\Delta \gg 2R$ becomes

$$-\ln V^{\rm perp}_{\rm exact} \xrightarrow{\Delta\gg 2R} \frac{3\pi}{5}\,\frac{d}{R}\,\frac{1+u^2+\sqrt{1+u^2}}{u^2}\,. \tag{30}$$

Two structural differences from the parallel-axis case:

- *Sign of the sub-leading phase correction.* Expanding the deep-saturated Eq. (30) in u,

  $$-\ln V^{\rm perp}_{\rm exact} \;=\; \frac{6\pi}{5}\,\frac{d}{R}\,u^{-2} \;+\; \frac{9\pi}{10}\,\frac{d}{R} \;+\; \mathcal{O}(u^2)\,, \tag{31}$$

  whose leading term is the perpendicular square law, and whose first correction has the *opposite sign* to the parallel-axis case (where the $1-(\Delta/d)^2$ factor reduces the exponent): parallel-axis is more permissive than its leading-order limit, perpendicular more suppressive.

- *No singularity at $\Delta = d$.* The perpendicular formula (29) is regular for all positive u, since the perpendicular arms never coincide with the partner particle; only the universal $\Delta \geq 2R$ bound applies. At QGEM-class $\Delta \gg 2R$ the exponent is large (Section 6) and $V^{\rm perp}_{\rm exact} \approx 0$, far from the $V \to 1$ blow-up of the parallel-axis formula.

4.3 Discrimination-feasible windows

A central consequence of the finite-size self-energy is that the residual-visibility window moves *out* of the QGEM operating regime. Because the saturated rate is set by d/R (with $R \sim$ 1–2 μm) rather than d/Δ, the exponent at QGEM-class $\Delta \lesssim d$ is enormous: $-\ln V \gtrsim 10^2$ in every geometry (Section 6), so $V \approx 0$ and is not directly resolvable. Defining the experimentally interesting window as $0.05 < V < 0.95$, the corrected leading saturated laws place it at $\Delta \gg d$

rather than $\Delta \sim d$; for silica ($\rho = 2200$ kg/m^3), $m = 10^{-14}$ kg, $d = 150$ μm,

$$\begin{aligned} \text{parallel-axis } (V_{\text{lead}}^{\text{par}}): &\quad \Delta/d \in [9.6,\ 73]\,, \\ \text{perpendicular } (V_{\text{lead}}^{\text{perp}}): &\quad \Delta/d \in [13.6,\ 104]\,, \\ \text{magic angle } (V_{\text{lead}}^{\text{magic}}): &\quad \Delta/d \in [3.9,\ 10.8]\,. \end{aligned} \tag{32}$$

Unlike the old cube law these windows are material-dependent (they scale with R, hence with $m^{1/3}$ at fixed ρ): for $m = 10^{-13}$ kg the parallel window shifts to $\Delta/d \in [6.5,\ 50]$. The magic-angle window sits at the smallest Δ/d because its steeper d^5/Δ^4 phase falloff offsets the saturated energy fastest. In all three cases the window requires the superposition arm to exceed the inter-particle separation by an order of magnitude or more — a regime no current QGEM design targets, and one in which the perpendicular layout (singularity-free) is the only one of the three that remains analytically controlled, since the parallel formula is restricted to $\Delta < d$.

4.4 Implication for QGEM design

At the QGEM design point ($m = 10^{-14}$ kg silica, $R = 1.03$ μm, $d = 150$ μm, $\Delta = d/2 = 75$ μm, so $u = 0.5$, $\Delta/2R = 36.5$), the four visibility exponents from Eqs. (16), (25), (21), and (29) are

$$\begin{aligned} -\ln V_{\text{lead}}^{\text{par}} &\approx 1.09 \times 10^3\,, & -\ln V_{\text{exact}}^{\text{par}} &\approx 8.16 \times 10^2\,, \\ -\ln V_{\text{lead}}^{\text{perp}} &\approx 2.18 \times 10^3\,, & -\ln V_{\text{exact}}^{\text{perp}} &\approx 2.58 \times 10^3\,, \end{aligned} \tag{33}$$

i.e. $V \approx 0$ to hundreds of decades in every case — roughly 40–90$\times$ larger in the exponent than the old point-mass cube-law estimate ($-\ln V_{\text{lead}}^{\text{par,cube}} = 4\pi \approx 12.6$ at $u = 0.5$), because the deep-saturated self-energy $1.2\,Gm^2/R$ far exceeds the point energy $Gm^2/\Delta = 2Gm^2/d$ when $\Delta \gg R$. The qualitative ordering survives — perpendicular suppresses more strongly than parallel, the exact phase makes parallel more permissive and perpendicular more suppressive — but at QGEM-class parameters all four predict total decoherence. Residual visibility therefore cannot be the discriminator at the nominal operating point; the experimental handle is instead the *scaling exponent* (square vs. cube in d/Δ), the *orientation dependence* (perpendicular twice the parallel exponent at fixed R), and the residual *mass trend* $m^{-1/3}$, all of which are read off from how V *changes* as the geometry is swept, not from its absolute value at one point.

Remark 4.1 (Experimental design). QGC and standard quantum gravity must be compared through their parametric dependence, not a single-point visibility. Standard quantum gravity predicts $V \approx 1$ independent of (Δ, d, R, θ); QGC predicts the strong, orientation- and size-dependent suppression of Eq. (33). The discrimination is carried entirely by that parametric structure, so a QGEM run must specify and scan the arm orientation and particle size.

5 Oblique-Arm Geometry and the Magic Angle

The one-parameter family of arm orientations θ to the inter-particle axis interpolates between the two cardinal geometries and contains an isolated angle at which the leading-order BMV phase vanishes identically, where the visibility scaling switches from square to quartic. Each particle's arms are displaced along the common unit vector $\hat{n} = (\cos\theta, \sin\theta, 0)$. The self-decoherence rate $\Gamma_{\text{self}}(\Delta)$ depends only on Δ and R (Section 2), so the entire θ-dependence enters through the witness time and the phase null is exact.

5.1 Branch distances and witness phase at arbitrary θ

For arms tilted by θ relative to the inter-particle axis, the four branch separations are

$$r_{LL} = r_{RR} = d\,, \qquad r_{\pm}(\theta) \;=\; \sqrt{d^2 \pm 2\,d\,\Delta\,\cos\theta + \Delta^2}\,. \tag{34}$$

At $\theta = 0$ these reduce to $r_{\pm} = d \pm \Delta$ (the parallel-axis case of Section 3); at $\theta = \pi/2$ they collapse to $r_+ = r_- = \sqrt{d^2 + \Delta^2}$ (the perpendicular case). The witness phase magnitude is

$$|\Delta\phi(\theta)| \;=\; \frac{Gm^2\tau}{\hbar}\left|\frac{2}{d} - \frac{1}{r_+(\theta)} - \frac{1}{r_-(\theta)}\right|, \tag{35}$$

the natural one-parameter extension of Eqs. (13) and (18).

5.2 Multipole expansion and the $P_2(\cos\theta)$ structure

Writing $u \equiv \Delta/d$ and expanding the two inverse branch distances to fourth order in u (the odd-in-u terms cancel between r_+ and r_- because of the $\pm$ displacement) yields

$$|\Delta\phi(\theta)| \;=\; \frac{Gm^2\tau}{\hbar\,d}\Big[\,|\,1 - 3\cos^2\theta\,|\,u^2 \;+\; \tfrac{1}{4}\,|\,3 - 30\cos^2\theta + 35\cos^4\theta\,|\,u^4 \;+\; \mathcal{O}(u^6)\Big]\,. \tag{36}$$

The angular factor in the leading u^2 term is $-2P_2(\cos\theta)$ where P_2 is the second Legendre polynomial: the BMV witness phase is a Newtonian dipole-dipole interaction at large d/Δ. Equation (36) reproduces $|\Delta\phi| = (2Gm^2\tau/\hbar d)\,u^2$ at $\theta = 0$ (parallel-axis leading-order, Eq. (14)) and $|\Delta\phi| = (Gm^2\tau/\hbar d)\,u^2$ at $\theta = \pi/2$ (perpendicular, Eq. (19)).

5.3 The magic angle: $\cos^2\theta_m = 1/3$

Definition 5.1 (Magic angle)**.** The magic angle is the unique zero of the leading u^2 coefficient $|1 - 3\cos^2\theta|$ of Eq. (36),

$$\cos^2\theta_m = \tfrac{1}{3} \quad\Longleftrightarrow\quad \theta_m = \arccos(1/\sqrt{3}) \approx 54.7356^\circ\,. \tag{37}$$

At θ_m the dipole contribution to the BMV phase vanishes and the next-order (quadrupole) term takes over. Its coefficient evaluates to

$$\tfrac{1}{4}\,|\,3 - 30/3 + 35/9\,| \;=\; \tfrac{1}{4}\cdot\tfrac{28}{9} \;=\; \tfrac{7}{9}\,, \tag{38}$$

so

$$|\Delta\phi(\theta_m)|_{\rm lead} \;=\; \frac{7}{9}\,\frac{Gm^2\tau}{\hbar\,d}\,(\Delta/d)^4\,. \tag{39}$$

The witness criterion $|\Delta\phi| = \pi/2$ now fixes

$$\tau_{\rm BMV}^{\rm magic,lead} \;=\; \frac{9\pi\hbar\,d^5}{14\,Gm^2\,\Delta^4}\,, \tag{40}$$

which is parametrically longer than the cubic-geometry witness times of Eqs. (15) and (20) by a factor $\propto (d/\Delta)^2$.

Combining with the saturated per-particle self-decoherence rate $\Gamma_{\rm self}(\Delta) = (Gm^2/\hbar)[6/(5R)-$

$1/\Delta]$ from Eq. (12) and the two-particle exponent $2\Gamma_{\rm self}\tau$, the visibility at the magic angle is

$$V_{\rm lead}^{\rm magic} \;=\; \exp\Big[-\frac{9\pi}{35}\,\frac{d^5(6\Delta-5R)}{\Delta^5 R}\Big] \;\xrightarrow{\Delta\gg 2R}\; \exp\Big[-\frac{54\pi}{35}\,\frac{d^5}{R\,\Delta^4}\Big], \tag{41}$$

a *quartic* suppression law $\propto (d/\Delta)^4(d/R)$ in the deep-saturated QGEM regime — one power of d/Δ softer than the old point-mass quintic, mirroring the cube→square shift of the cardinal geometries (Section 3) — yet still qualitatively distinct from the square-law scaling of the two cardinal geometries. No CSL or Diósi–Penrose model can reproduce this orientation dependence, because in those models the decoherence rate depends only on the local mass distribution and is blind to the arm orientation; the magic-angle softening (quartic at θ_m versus square at the cardinal angles) is a direct fingerprint of the geometric structure of the Newtonian phase that QGC competes against.

The general one-parameter family follows from Eq. (36) at leading order, with the saturated rate (deep-saturated, $\Delta \gg 2R$):

$$V_{\rm lead}(\theta) \;=\; \exp\Big[-\frac{(6\pi/5)\,(d/\Delta)^2\,(d/R)}{|1-3\cos^2\theta|}\Big], \tag{42}$$

which reproduces $V_{\rm lead}^{\rm par}$ at $\theta = 0$ (denominator $|1-3| = 2$, giving prefactor $3\pi/5$), $V_{\rm lead}^{\rm perp}$ at $\theta = \pi/2$ (denominator $|1-0| = 1$, prefactor $6\pi/5$), and diverges as $\theta \to \theta_m$ where the dipole vanishes and the leading-order approximation must be replaced by Eq. (41). The exponent of Eq. (42) is strictly monotone in θ on each of the two intervals $[0, \theta_m)$ and $(\theta_m, \pi/2]$: the magic angle is the worst-case orientation for residual visibility (best-case for QGC discrimination), with the cardinal geometries straddling it. The orientation dependence itself — the $|1-3\cos^2\theta|$ in the denominator, inherited from the BMV phase's $P_2(\cos\theta)$ structure — is unaltered by the finite-size correction, because $\Gamma_{\rm self}$ is geometry-independent; it is the robust signature that no collapse model reproduces.

5.4 Exact-phase magic-angle visibility

Equation (41) retains only the leading u^4 term of the phase. The exact-phase calculation at θ_m proceeds without expansion. Defining the dimensionless function

$$g_m(u) \;\equiv\; \Big|\,2-\big(1+(2/\sqrt{3})\,u+u^2\big)^{-1/2}-\big(1-(2/\sqrt{3})\,u+u^2\big)^{-1/2}\Big|, \tag{43}$$

the exact magic-angle witness phase is $|\Delta\phi(\theta_m)| = (Gm^2\tau/\hbar d)\,g_m(u)$. Expanding $g_m(u)$ analytically gives the closed series (see Appendix B)

$$g_m(u) \;=\; \frac{7}{9}\,u^4\Big[1-\tfrac{4}{7}\,u^2+\mathcal{O}(u^4)\Big], \tag{44}$$

in which the u^2 (dipole) coefficient is identically zero — the defining property of the magic angle — and the u^4 coefficient is exactly $7/9$, recovering Eq. (39). The first non-trivial correction enters at u^6 with relative coefficient $-4/7$.

Setting $|\Delta\phi(\theta_m)| = \pi/2$ fixes the exact witness time $\tau_{\rm BMV}^{\rm magic,exact} = (\pi\hbar d/2Gm^2)/g_m(u)$; combining with the saturated rate $\Gamma_{\rm self}(\Delta) = (Gm^2/\hbar)[6/(5R) - 1/\Delta]$ and the two-particle exponent $2\Gamma_{\rm self}\tau$ (the Gm^2 cancels), the exact-phase visibility exponent is

$$-\ln V_{\rm exact}^{\rm magic} = \frac{\pi d}{g_m(u)}\left[\frac{6}{5R}-\frac{1}{\Delta}\right], \qquad V_{\rm exact}^{\rm magic} = \exp\!\Big[-\pi d\big(\tfrac{6}{5R}-\tfrac{1}{\Delta}\big)/g_m(u)\Big]. \tag{45}$$

In the deep-saturated regime $\Delta \gg 2R$, $[6/(5R) - 1/\Delta] \to 6/(5R)$ and substituting Eq. (44) yields the analytic expansion

$$-\ln V_{\rm exact}^{\rm magic} \xrightarrow{\Delta\gg 2R} \frac{54\pi}{35}\frac{d^5}{R\,\Delta^4} + \frac{216\pi}{245}\frac{d^3}{R\,\Delta^2} + \mathcal{O}(d/R)\,. \tag{46}$$

The leading term reproduces the deep-saturated quartic of Eq. (41). The decisive qualitative feature is the *sign* of the sub-leading correction: the $216\pi/245$ coefficient is positive (it inherits the $+36/49$ of the phase series and the $6/(5R)$ of the rate, $(9/7)\times(6/5) = 54/35$ leading, $(36/49)\times(6/5) = 216/245$ sub-leading), so

$$V_{\rm exact}^{\rm magic} < V_{\rm lead}^{\rm magic} \quad \text{for all } u > 0\,. \tag{47}$$

The exact phase makes the magic-angle suppression *stronger* than the leading quartic. This is the same direction as the perpendicular geometry (Eq. (29) versus Eq. (21)) and the opposite direction to the parallel-axis geometry (Eq. (25) versus Eq. (16)), completing the exact-vs-leading trilogy across the three computed geometries.

5.5 Numerical regime and discrimination window

Unlike the old point-mass quintic, the corrected exponent depends on the particle size R and is therefore material- and d-specific. Representative values for silica ($\rho = 2200$ kg/m^3), $m = 10^{-14}$ kg ($R = 1.03$ μm), $d = 150$ μm:

$$\begin{aligned} -\ln V_{\rm exact}^{\rm magic}(u=0.5) &= 1.33\times10^4 \quad (V\approx 0)\,,\\ -\ln V_{\rm exact}^{\rm magic}(u=1.0) &= 1.57\times10^3 \quad (V\approx 0)\,,\\ -\ln V_{\rm exact}^{\rm magic}(u=2.0) &= 5.38\times10^2 \quad (V\approx 3\times10^{-234})\,, \end{aligned} \tag{48}$$

the corresponding leading deep-saturated quartic exponents being 1.12×10^4, 7.04×10^2, and 44.1. The exact-versus-leading gaps (2.09×10^3, 863, 494) all have the same sign (exact more suppressive than leading), consistent with the positive $(216\pi/245)d^3/(R\Delta^2)$ correction of Eq. (46). At these QGEM-class separations $V\approx 0$ for every $u \lesssim 2$, as for the cardinal geometries.

Because the exponent now carries the explicit d/R factor, the residual-visibility window $0.05 < V < 0.95$ moves to $\Delta \gg d$ and is material-dependent. Computed numerically from Eq. (45) for the same silica, $m = 10^{-14}$ kg, $d = 150$ μm:

$$\text{magic-angle (leading saturated):} \qquad \Delta/d \in [3.9,\ 10.8]\,, \tag{49}$$

sitting at smaller Δ/d than the cardinal windows of Eq. (32) ($[9.6, 73]$ parallel, $[13.6, 104]$ perpendicular at the same parameters) because its steeper d^5/Δ^4 phase falloff offsets the saturated energy fastest. In all three geometries the window requires $\Delta > d$; the discrimination at QGEM-class $\Delta \lesssim d$ is carried not by a residual-visibility plateau but by the *scaling exponent* (quartic at θ_m versus square at the cardinal angles), the orientation dependence, and the mass trend.

5.6 Near-magic anti-resonance and robustness to experimental jitter

A real apparatus holds neither the arm orientation nor the arm length exactly; the magic-angle structure survives both, with one sharp new feature in the near-magic neighbourhood.

The one-sided anti-resonance. Write $\varepsilon \equiv \cos^2\theta - 1/3$ for the detuning from the magic angle. For $\varepsilon > 0$ (the parallel side) the residual dipole term $3\varepsilon u^2$ and the quadrupole term $\frac{7}{9}u^4$ of the phase enter with opposite signs, and the exact phase possesses a *true zero* at

$$u_0(\varepsilon) \;=\; \sqrt{27\varepsilon/7}\,[1+\mathcal{O}(\varepsilon)]\;: \tag{50}$$

the witness phase vanishes there, the witness time diverges, and the visibility at the witness criterion crashes to zero. The dip is deep across its whole width: at the leading-order location $u_* = \sqrt{27\varepsilon/7}$, where the residual phase is $(243/343)\,\varepsilon^3$, the decoherence exponent already exceeds $2.26\,\varepsilon^{-7/2}$ ($\approx 10^5$ at $\varepsilon = 0.05$). For $\varepsilon < 0$ (the perpendicular side) the two terms carry the same sign and no cancellation occurs—the anti-resonance is one-sided. A 1° misalignment toward the parallel side gives $\varepsilon = \sin(2\theta_m)\,\Delta\theta \approx 0.016$ and $u_* \approx 0.25$, inside the QGEM design range: an experimenter scanning u at fixed near-magic θ would see the visibility crash at this specific u_*, and the crash location measures the misalignment through Eq. (50)—an independent in-situ calibration of arm orientation. No collapse model can produce a one-sided, orientation-locked anti-resonance, because none has an analogue of the dipole–quadrupole interference behind it.

Angular jitter. Modelling the orientation as Gaussian, $\theta \sim \mathcal{N}(\theta_m, \sigma_\theta^2)$, the detuning is Gaussian with $\sigma_\varepsilon = (2\sqrt{2}/3)\,\sigma_\theta$. The jitter contaminates the magic-angle average only when a 1σ excursion reaches the anti-resonance trigger, at

$$\sigma_\theta^{\mathrm{AR}}(u) \;=\; \frac{7}{18\sqrt{2}}\,u^2\ \mathrm{rad} \;\approx\; 15.8\,u^2\ \mathrm{deg}\,; \tag{51}$$

at 1° alignment the magic-angle quartic is protected for $u \gtrsim 0.25$—all QGEM-class operating points—and at 0.1° for $u \gtrsim 0.08$. Below threshold the averaged exponent inflates only perturbatively, $\langle E\rangle/E_{\mathrm{magic}} \approx 1 + (648/49)\,\sigma_\theta^2/u^4$: for 1° jitter, 0.4% at $u = 1$ and 6.4% at $u = 0.5$. The quartic prediction is not a knife-edge.

Arm-length jitter. Percent-level separation jitter σ_Δ/Δ is benign in a structurally different way: the averaged exponent correction is $\alpha_2(u)\,(\sigma_\Delta/\Delta)^2$ with $\alpha_2(u) = (105+24u^2)/(7+4u^2) \le 15$—bounded and slowly varying in u, in contrast to the $1/u^4$ growth of the angular channel. By convexity, sub-percent Δ-jitter *enhances* the ensemble-averaged visibility below $u \approx 1.8$ and suppresses it above; the sign flip is itself an operational diagnostic tied to the magic-angle phase structure. The anti-resonance dip of Eq. (50) has half-width $u_*/(2F)$ at suppression target F, so 1% arm-length control—routine for QGEM-class designs—resolves the dip core up to targets $F \approx 50$; probing deeper into the dip requires proportionally finer control.

5.7 Experimental and theoretical takeaways

Equations (42) and (45) extend the QGC prediction of Section 3 from a binary choice between two cardinal arm orientations to a continuous one-parameter family with three qualitatively distinct regimes:

- *Square regime*, $|\cos^2\theta - 1/3|$ bounded away from zero: in the deep-saturated regime $\Delta \gg 2R$ the visibility scales as $\exp[-c(\theta)(d/\Delta)^2(d/R)]$ with $c(\theta) = (6\pi/5)/|1-3\cos^2\theta|$, interpolating smoothly between $3\pi/5$ (parallel) and $6\pi/5$ (perpendicular).
- *Quartic regime*, $\theta = \theta_m$: visibility scales as $\exp[-(54\pi/35)(d/\Delta)^4(d/R)]$, with the dipole BMV phase contribution exactly cancelled by the geometric condition $\cos^2\theta = 1/3$ and the quadrupole taking over.
- *Transition region*, $|\cos^2\theta - 1/3| \lesssim u^2$: the leading-order expansion breaks down and the exact formula $V = \exp[-\pi d(\frac{6}{5R} - \frac{1}{\Delta})/g(u,\theta)]$ must be used; the discrimination window widens monotonically as $\theta \to \theta_m$.

An apparatus that rotates its arms relative to the inter-particle axis sweeps three distinct QGC signatures in one platform—four, counting the one-sided near-magic anti-resonance of Section 5.6, which doubles as an in-situ orientation calibration. The $\cos^2\theta_m = 1/3$ phase null is itself a test of Newtonian gravity at mesoscopic separations: any theory whose phase response retains a non-zero dipole at the magic angle (any modification breaking P_2 structure) gives residual BMV signal where QGC and Newtonian gravity predict the quartic-suppressed null. A combined measurement at $\theta = 0$, $\theta = \pi/2$, and $\theta = \theta_m$ tests three independent QGC predictions with the same hardware: square suppression with prefactor $3\pi/5$, square suppression with prefactor $6\pi/5$, and quartic suppression with prefactor $54\pi/35$.

The orientation dependence is preserved exactly at every angle. Because the finite-size self-decoherence rate $\Gamma_{\text{self}}(\Delta)$ is geometry-independent (it depends only on Δ and R), the entire θ-dependence of $V(\theta)$ comes from the witness time $\tau_{\text{BMV}}(\theta)$; the $P_2(\cos\theta)$ structure and the magic-angle null are therefore robust against the finite-size correction. The residual mass dependence is the weak $m^{-1/3}$ trend (through R) of Section 3, identical at every angle, in place of the old strict mass-independence.

6 Numerical Predictions and Comparison with Competing Theories

6.1 Forecast for QGEM-class parameters

The QGEM proposal [3] considers two equal masses in Stern–Gerlach interferometers; we adopt silica spheres ($\rho = 2200$ kg/m^3) with $m \sim 10^{-14}$–10^{-13} kg (radius $R \sim 1$–2 μm), arm separations $\Delta \sim 60$–250 μm, and inter-particle separations $d \sim 120$–200 μm, giving a nominal design ratio $\Delta/d \approx 0.5$ (i.e. $\Delta = d/2$) and $\Delta/2R \approx 14$–195 — deep in the spatially saturated regime. The corrected prediction depends on the particle size R as well as on Δ/d and the arm-to-axis angle, so we fix the silica reference material and tabulate the leading-saturated visibility versus Δ/d at $m = 10^{-14}$ kg, $d = 150$ μm ($R = 1.03$ μm) in Table 1. All entries follow from Eqs. (16), (21), and (41).

Table 1. *QGC-predicted residual visibility at the BMV witness criterion $\phi = \pi/2$, as a function of Δ/d, in the three computed geometries, for silica ($\rho = 2200$ kg/m^3), $m = 10^{-14}$ kg ($R = 1.03$ μm), $d = 150$ μm (leading-saturated formulae). Unlike the old cube law, the values are material- and d-dependent ($\propto m^{-1/3}$ at fixed ρ). "$\theta = \theta_m$" is the magic angle* $\arccos(1/\sqrt{3}) \approx 54.7°$*; "QG std." is the standard quantum-gravity prediction (no decoherence: $V = 1$). Entries below $\sim 10^{-50}$ are written ~ 0. At the QGEM design point $\Delta/d = 0.5$ all three give $V \approx 0$; the discrimination-feasible window ($0.05 < V < 0.95$) sits at $\Delta \gg d$ (Eq. (32)).*

Δ/d	$\theta = 0$ (parallel)	$\theta = \pi/2$ (perp.)	$\theta = \theta_m$ (magic)	QG std.
0.50	~ 0	~ 0	~ 0	1
1.00	~ 0	~ 0	~ 0	1
2.00	1.6×10^{-30}	~ 0	7.1×10^{-20}	1
5.00	1.7×10^{-5}	2.8×10^{-10}	0.32	1
10.0	0.064	4.1×10^{-3}	0.93	1
20.0	0.50	0.25	1.00	1
40.0	0.84	0.71	1.00	1
73.0	0.95	0.90	1.00	1
104	0.97	0.95	1.00	1

The QGEM nominal design $\Delta = d/2$ gives total decoherence in all three geometries ($V \approx 0$; Table 1, top rows): the finite-size self-energy $\approx 1.2\,Gm^2/R$ is so large at $R \sim 1$ μm that the visibility exponent reaches $\sim 10^3$. The discrimination-feasible window therefore requires moving to $\Delta \gg d$, where the arm separation is large enough that the square/quartic suppression relaxes toward unity:

- the perpendicular-axis layout reaches the window $0.05 < V < 0.95$ at $\Delta/d \in [13.6, 104]$ from Eq. (21), with a clear *square-law* signature $-\ln V \propto (d/\Delta)^2(d/R)$ in the suppression as Δ/d varies;
- the parallel-axis layout reaches it at $\Delta/d \in [9.6, 73]$ from Eq. (16) (the exact-phase form is restricted to $\Delta < d$ and is itself $V \approx 0$ there);
- the magic-angle layout reaches it at the smallest $\Delta/d \in [3.9, 10.8]$ from Eq. (41), following the *quartic* law $-\ln V \propto (d/\Delta)^4(d/R)$ whose steeper falloff offsets the saturated energy fastest.

These windows scale with $R \propto m^{1/3}$ (e.g. the parallel window shifts to $\Delta/d \in [6.5, 50]$ at $m = 10^{-13}$ kg), a direct, testable consequence of the finite-size self-energy that the old mass-independent cube law lacked.

6.2 Comparison with competing theories

Five candidate predictions for the BMV/QGEM visibility, evaluated at the same parameters:

QGC and standard quantum gravity differ qualitatively in their *parametric* structure: standard QG predicts $V \approx 1$ independent of all of (Δ, d, R, θ), while QGC predicts a strong, orientation- and size-dependent suppression whose exponent follows a square law $(d/\Delta)^2(d/R)$ in the cardinal geometries and a quartic $(d/\Delta)^4(d/R)$ at the magic angle. Against the collapse models, the QGC prediction is parameter-free (no analogue of the CSL collapse rate λ or the Diósi–Penrose smearing σ_0) and carries only the weak residual $m^{-1/3}$ trend, in place of the $\propto m^2$ of Diósi–Penrose, the local-density dependence of CSL, or the temperature/composition dependence of Pikovski et al. Decisively, none of the collapse models has any analogue of

Table 2. *Predictions for the BMV visibility from competing theoretical frameworks, at the QGEM nominal design point. "V" is the residual modulus of the off-diagonal density-matrix element at the witness criterion.*

Framework	V at QGEM design	Distinctive scaling
Standard QG (no decoherence)	≈ 1	none
QGC (constrained-IF, this work)	≈ 0	$(d/\Delta)^2(d/R)$ (square; quartic at magic); residual $m^{-1/3}$; orientation-dep.
Diósi–Penrose [5, 7]	$\exp(-\Gamma_{DP}\,\tau_{\rm BMV})$	$\propto m^2$, smearing σ_0
CSL [8]	$\exp(-\lambda_{\rm CSL}\,\tau_{\rm BMV}\,N)$	free λ, density-dependent
Pikovski et al. [9]	$\exp(-\Gamma_{\rm PB}\,\tau_{\rm BMV})$	$\propto T\times$ (composition)
Classical (LOCC) gravity	1 but $C_{AB}=0$	no entanglement at all

the orientation-dependent square→quartic crossover and the magic-angle phase null: their decoherence rates depend only on the local matter distribution and are blind to the arm orientation relative to the inter-particle axis. The gravity-collapse landscape itself is narrowing around the Diósi–Penrose class: the generalised Károlyházy model has recently been experimentally excluded by an underground spontaneous-radiation search [17], while the Diósi–Penrose phenomenology admits complementary formal developments, including a quantum-filtering (stochastic-master-equation) formulation of its collapse dynamics [18].

6.3 Falsification routes

The experimental discrimination is binary at the QGEM design point, where QGC predicts total decoherence:

- If QGEM observes the standard BMV entanglement amplitude ($V \approx 1$) at $\Delta/d \sim 1$ in any of the three computed arm orientations, QGC's constrained-IF picture is *falsified.*
- If QGEM observes the strong gravitational self-decoherence that QGC predicts, the discriminating signatures appear in the *parametric sweep*: a *square-law* suppression $-\ln V \propto (d/\Delta)^2(d/R)$ in the cardinal geometries with the geometry-dependent prefactor (parallel $3\pi/5$, perpendicular $6\pi/5$), together with the residual $m^{-1/3}$ mass trend at fixed (d, Δ, ρ), selects QGC over Diósi–Penrose, CSL, and Pikovski-class models.
- If, in addition, QGEM observes a *quartic-law* suppression at the magic angle $\theta_m = \arccos(1/\sqrt{3})$ with prefactor $54\pi/35$ and the $\cos^2\theta = 1/3$ phase null, this rules out any decoherence mechanism whose rate is independent of the arm orientation relative to the inter-particle axis – a constraint that no collapse model can satisfy.

Failure modes that would not falsify QGC specifically: any visibility loss with explicit m^2 scaling (consistent with any collapse model), or any temperature dependence (consistent with Pikovski et al.). Both are controlled by the standard experimental sweeps QGEM already plans, and the QGC mass trend ($m^{-1/3}$) is qualitatively opposite to the collapse-model m^2, so a mass scan alone separates them.

6.4 Reproducibility

The numerical values in Tables 1 and 2 follow directly from the finite-size saturated self-energy $E_G^{\rm self}(\Delta)$ of Eq. (10) and the six QGC visibility formulae — cardinal-square in both arm orientations, exact-phase corrections in both, the magic-angle quartic, and the exact magic-angle form — evaluated at the silica reference parameters with the standard BMV phase formula. Each headline scaling is recovered analytically: the saturation $E_G^{\rm self} \to 1.2\,Gm^2/R$ with C^1 continuity at $\Delta = 2R$, the square law, the residual $m^{-1/3}$ trend, the leading-to-exact crossover, the $P_2(\cos\theta)$ angular dependence, the magic-angle phase null, the quartic prefactor $54\pi/35$, and the $7/5$-enhanced cube/quintic edge at $\Delta \to 2R^+$.

7 Discussion and Conclusions

7.1 Summary of results

The QGC visibility prediction for BMV/QGEM, across the full one-parameter family of arm orientations θ, is driven by the *finite-size* Diósi–Penrose branch-pair self-energy, which saturates at $E_G^{\rm self} \to (6/5)Gm^2/R$ for a solid sphere of radius R once $\Delta \geq 2R$ (Section 2); QGEM operates deep in this saturated regime ($\Delta \gg 2R$). In the two cardinal geometries ($\theta = 0$ parallel, $\theta = \pi/2$ perpendicular) and at the magic angle ($\cos^2\theta_m = 1/3$), the deep-saturated visibility exponents are

$$\begin{aligned} -\ln V_{\rm lead}^{\rm par} &= \tfrac{3\pi}{5}\,(d/\Delta)^2\,(d/R)\,, \\ -\ln V_{\rm lead}^{\rm perp} &= \tfrac{6\pi}{5}\,(d/\Delta)^2\,(d/R)\,, \\ -\ln V_{\rm lead}^{\rm magic} &= \tfrac{54\pi}{35}\,(d/\Delta)^4\,(d/R)\,, \end{aligned} \tag{52}$$

with exact-phase corrections (Eqs. (25), (29), (45)) that carry the explicit factor $[6/(5R) - 1/\Delta]$ in place of the old $1/\Delta$, and the angular interpolation $-\ln V_{\rm lead}(\theta) = (6\pi/5)(d/\Delta)^2(d/R)/|1 - 3\cos^2\theta|$ between the cardinal geometries (Eq. (42)). These carry a residual $m^{-1/3}$ mass dependence (through R) and are otherwise free of fitted parameters. The leading prefactor doubles between the cardinal square-law geometries; the sign of the next-to-leading phase correction flips between them; and at the magic angle the dipole BMV phase vanishes identically and the suppression becomes quartic with prefactor $54\pi/35$, exact-phase strictly stronger than leading.

7.2 Origin of the square law

Two physical scales control the BMV experiment. The phase scale is the gravitational interaction energy difference between branches, $\Delta E \sim Gm^2\Delta^2/d^3$ (for $\Delta \ll d$ in either geometry, up to the geometry-dependent prefactor); it sets the witness time $\tau \sim \hbar/\Delta E \sim \hbar d^3/(Gm^2\Delta^2)$. The decoherence scale is the per-particle self-decoherence rate, which for a finite-size sphere saturates at $\Gamma_{\rm self} \to (6/5)\,Gm^2/(\hbar R)$ — set by the particle *size* R, not by Δ. Their product

$$\Gamma_{\rm self}\,\tau \;\sim\; \frac{Gm^2}{\hbar R}\cdot\frac{\hbar d^3}{Gm^2\Delta^2} \;=\; \Big(\frac{d}{\Delta}\Big)^2\,\frac{d}{R} \tag{53}$$

gives the square law. The Gm^2 cancels, but the R in $\Gamma_{\rm self}$ survives and, at fixed density $R \propto m^{1/3}$, leaves the residual $m^{-1/3}$ trend. The square power (one softer than the point-mass cube) is therefore an algebraic consequence of the self-energy saturation that QGC identifies; it is not a free choice. Only at the unphysical edge $\Delta \to 2R^+$, where the branches just cease to overlap and $\Gamma_{\text{self}} \to (7/5)Gm^2/(\hbar\Delta)$, does the cube law re-emerge — with its coefficient enhanced by 7/5 relative to the published point-mass values.

7.3 What is, and is not, derived

The square-law and quartic-law forms of the visibility, the leading and exact-phase prefactors, the magic-angle identification, and the $P_2(\cos\theta)$ angular interpolation are derived from QGC's constrained Feynman–Vernon influence functional plus the standard Newtonian phase formula expanded in the Legendre generating function for the branch distances (Appendix B). The finite-size saturated self-energy $E_G^{\text{self}}(\Delta)$ of Eq. (10) — the uniform-sphere Diósi–Penrose branch-pair energy — is computed in closed form and verified numerically against Monte-Carlo integration of the source-pair integral. The dressed-coherent-state overlap structure of the gravitational field, the temporal saturation of the overlap exponent at $\xi\ \ln(d/\ell_P)$, and the per-particle linear-growth rate are inputs from the canonical core paper [19] on the constrained influence functional; its single-mass scale $E_G = GM^2/d$ is the point-particle limit of E_G^{self}, recovered when the source is treated as point-like. The energy scale E_G is established (it is the classical branch-pair self-energy); the identification of the linear self-decoherence *rate* with $E_G/\hbar$ at first order in G is the core paper's central conjecture, motivated by the linearised Hamiltonian constraint but not yet established at the operator level. Every prediction below inherits that conditional status: the visibility law is the consequence QGC carries into the BMV setting if its G^1 rate is correct, and its experimental test is a test of that rate.

Beyond the conditional status of the G^1 rate itself, two further ingredients are conjectured rather than derived:

- *Validity beyond linearised gravity.* The derivation uses the Newtonian limit of linearised general relativity throughout. At QGEM-class parameters this is well within the regime of validity (graviton wavelength $\gg$ particle separation), but extension to higher mass scales would require post-Newtonian corrections to both the BMV phase formula and the constrained-IF dressed-state overlap.

- *Effect of imperfect coherent-state alignment.* We have assumed that the gravitational field for each matter configuration is exactly in the constraint-enforced dressed coherent state. Departures from exact alignment (due to, e.g., decoherence of the gravitational field by external sources) would weaken the QGC prediction; the magnitude of this effect has not been estimated.

7.4 Limitations and caveats

Two saturations: the prediction requires both the *temporal* saturation ($\tau_{\text{BMV}} > d/c$, the gravitational sound-crossing time; at QGEM scales $d/c \sim 10^{-12}$ s versus $\tau_{\text{BMV}} \sim 1$ s, satisfied by twelve orders of magnitude) and the *spatial* saturation ($\Delta \geq 2R$, the branches no longer overlapping; at QGEM $\Delta/2R \sim 14$–195, deeply satisfied). The point-mass cube law would re-emerge only in the unphysical edge $\Delta \to 2R^+$, never realised by a rigid sphere at QGEM separations.

Visibility versus concurrence: we have computed the visibility of the off-diagonal element of the two-particle reduced density matrix, not the full two-qubit concurrence. For the BMV witness criterion, the off-diagonal modulus dominates the concurrence monotonically; the square law therefore translates directly to a concurrence suppression. A full treatment of the four-by-four matter density matrix is straightforward and would refine the numerical prefactors by $\mathcal{O}(1)$ but does not alter the square-law scaling or the $m^{-1/3}$ trend.

Environmental decoherence: any non-gravitational decoherence (residual gas, blackbody radiation, internal degrees of freedom) [14] acts in parallel and adds to the QGC suppression. Because the QGC self-decoherence exponent is large ($\gtrsim 10^2$) at QGEM-class $\Delta \lesssim d$, distinguishing the gravitational signature from the background requires the parameter sweep across Δ/d, particle size R, and orientation θ that the square/quartic laws predict, rather than a single-point visibility measurement.

7.5 Relation to other master-equation derivations

Several master equations for gravitationally induced decoherence have been derived from quantising linearised gravity perturbatively. Anastopoulos and Hu [10] and Blencowe [11] obtain a Lindblad equation by tracing over the transverse-traceless graviton bath; the resulting rate is $\mathcal{O}(G^2)$ in the matter–graviton coupling and depends on the temperature of the graviton bath. The QGC rate (12) is $\mathcal{O}(G)$ in the same coupling because the constrained influence functional retains the Hamiltonian-constraint sector of the gravitational field, which the perturbative trace integrates out [19]. At QGEM-class parameters the QGC self-decoherence rate exceeds the Anastopoulos–Hu/Blencowe rate by many orders of magnitude (the QGC square-law suppression would be invisible in either of the perturbative treatments), so an observation of strong, geometry-dependent suppression at the QGEM design point would also discriminate between QGC and these perturbative master equations.

7.6 Place among other near-term tests of QGC

This prediction sits alongside two other QGC-specific experimental signatures, summarised in Table 3.

Table 3. *QGC-specific experimental discriminators among near-term proposals.*

Test	Observable	Discriminates QGC from
Single-mass decoherence [19]	$\Gamma_{\text{dec}} = GM^2/(\hbar d)$ (point limit)	no decoherence
Single-mass entanglement decay [20]	$C_{AB}(t) = C_{AB}(0)e^{-GM^2 t/\hbar d}$	no decoherence
This work (BMV/QGEM)	$-\ln V \propto (d/\Delta)^2(d/R)$; $m^{-1/3}$	no decoherence, CSL, Diósi–Penrose, Pikovski

The single-mass tests [19, 20] discriminate QGC from theories that predict no gravitational decoherence at all; they cannot distinguish QGC from collapse models that also predict mass-dependent decoherence. Their headline scale $E_G = GM^2/d$ is the point-particle limit of the finite-size self-energy used here (Eq. (10)); a precise finite-size BMV prediction requires the *saturated* self-energy $\approx 1.2\,Gm^2/R$, which differs from the point form whenever the source size R is not negligible against the relevant separation. The BMV test is the near-term proposal whose orientation-dependent square/quartic scaling and magic-angle phase null are incompatible with all extant collapse models simultaneously, regardless of the (now weak, $m^{-1/3}$) mass trend.

7.7 Does entanglement certify quantum gravity? The self-decoherence caveat

The BMV/QGEM programme rests on the LOCC argument that gravity-mediated entanglement certifies a non-classical gravitational field [2, 4]. A growing literature questions whether the observation of entanglement is by itself sufficient evidence for quantised spacetime: Aziz and Howl [16] show that classical (or effective/semiclassical) theories of gravity coupled to quantum matter can themselves produce gravitationally induced entanglement, so the observation of entanglement need not certify a quantised gravitational field. Our analysis adds a complementary, internal caveat that sharpens the practical stakes: within QGC, the same finite-size self-energy also self-decoheres each particle independently, and at QGEM-class parameters this self-decoherence swamps the mutual entangling channel. The self-energy $\approx 1.2\, Gm^2/R$ exceeds the mutual entangling energy $\Delta E_{\rm BMV} \sim Gm^2\Delta^2/d^3$ by the large factor $\sim 0.6\, d^3/(R\Delta^2)$, so the witness coherence is lost long before the mutual entanglement is half-formed. This does not weaken the LOCC logic where entanglement *is* observed; rather, it predicts that within QGC the entanglement is not observable at the nominal design point at all, and it relocates the experimentally accessible QGC signature to the parametric structure of the self-decoherence (square/quartic scaling, orientation dependence, $m^{-1/3}$ trend) rather than to the entanglement amplitude itself. We emphasise that the framework's canonical single-mass energy scale $E_G = GM^2/d$ (Paper I) is the point-particle limit; a precise finite-size BMV prediction — the central correction of this paper — requires the saturated Diósi–Penrose self-energy of Eq. (10).

7.8 Conclusions

QGC predicts that gravitational self-decoherence of the source masses suppresses the BMV/QGEM entanglement visibility, with a form that depends on the arm-to-axis angle θ and — through the finite particle size R — on the mass. For solid spheres in the deep-saturated regime $\Delta \gg 2R$ that QGEM occupies, the suppression in the two cardinal geometries is a *square law* $-\ln V \propto (d/\Delta)^2(d/R)$ with prefactor $3\pi/5$ (parallel) or $6\pi/5$ (perpendicular); at the magic angle $\cos^2\theta_m = 1/3$ the dipole BMV phase vanishes identically and the suppression becomes *quartic*, $-\ln V \propto (54\pi/35)(d/\Delta)^4(d/R)$. The old point-mass cube/quintic laws survive only at the unphysical edge $\Delta \to 2R^+$, with coefficients enhanced by 7/5. Including the exact phase, the parallel-axis formula is more permissive than its leading-order limit while the perpendicular and magic-angle formulae are more suppressive. At the QGEM design point $\Delta = d/2$ all three predictions give $V \approx 0$; the discrimination-feasible regime requires $\Delta \gg d$ (parallel $\Delta/d \in [9.6, 73]$, perpendicular $[13.6, 104]$, magic $[3.9, 10.8]$ for silica, $m = 10^{-14}$ kg, $d = 150$ μm), and these windows themselves scale as $m^{1/3}$. Observation of standard BMV entanglement at QGEM-class parameters would falsify QGC's constrained-IF picture. The robust QGC discriminators that survive the finite-size correction — and that no collapse model reproduces — are the *orientation dependence* (the $P_2(\cos\theta)$ interpolation and the magic-angle phase null), the *scaling exponent* (square versus the collapse-blind constancy; quartic at the magic angle), and the *mass trend* ($m^{-1/3}$, opposite in sign to the m^2 of every collapse model); together they select QGC in a single multi-parameter experiment.

Appendix

A Algebra of the Exact-Phase Visibility Formulae

The algebraic steps from the exact branch energies of Section 2 to the exact-phase visibility formulae of Section 4, with $u \equiv \Delta/d$ throughout.

A.1 Parallel-axis arms: from energy to visibility

The four-branch interaction energies for the parallel-axis layout are

$$E_{LL} = E_{RR} = -\frac{Gm^2}{d}\,, \qquad E_{LR} = -\frac{Gm^2}{d-\Delta}\,, \qquad E_{RL} = -\frac{Gm^2}{d+\Delta}\,. \tag{54}$$

The witness phase difference at time τ is

$$\Delta\phi_{\rm par}(\tau) = \frac{\tau}{\hbar}(E_{LL} + E_{RR} - E_{LR} - E_{RL}) = -\frac{Gm^2\tau}{\hbar}\left[\frac{2}{d} - \frac{1}{d-\Delta} - \frac{1}{d+\Delta}\right]. \tag{55}$$

Algebraically,

$$\frac{2}{d} - \frac{1}{d-\Delta} - \frac{1}{d+\Delta} \;=\; \frac{2(d^2-\Delta^2) - d(d+\Delta) - d(d-\Delta)}{d(d^2-\Delta^2)} \;=\; -\frac{2\Delta^2}{d(d^2-\Delta^2)}\,, \tag{56}$$

so

$$|\Delta\phi_{\rm par}(\tau)| \;=\; \frac{2\,Gm^2\,\Delta^2\,\tau}{\hbar\,d(d^2-\Delta^2)}\,. \tag{57}$$

Setting this equal to $\pi/2$ (the BMV witness criterion) gives

$$\tau_{\rm BMV}^{\rm par,exact} \;=\; \frac{\pi\hbar\,d(d^2-\Delta^2)}{4\,Gm^2\,\Delta^2} \;=\; \frac{\pi\hbar}{4Gm^2}\,\frac{d^3}{\Delta^2}\,(1-u^2)\,. \tag{58}$$

The leading-order limit ($u \to 0$) gives $\tau_{\rm BMV}^{\rm par,lead} = \pi\hbar d^3/(4Gm^2\Delta^2)$ as in Eq. (15).

The QGC visibility uses the saturated finite-size self-decoherence rate $\Gamma_{\rm self}(\Delta)$ of Eq. (12) (valid for $\Delta \geq 2R$) and the two-particle exponent $2\Gamma_{\rm self}\tau$; the Gm^2 cancels:

$$\begin{aligned} -\ln V_{\rm exact}^{\rm par} \;&=\; 2\Gamma_{\rm self}(\Delta)\,\tau_{\rm BMV}^{\rm par,exact} \;=\; 2\,\frac{Gm^2}{\hbar}\Big[\frac{6}{5R} - \frac{1}{\Delta}\Big]\cdot\frac{\pi\hbar\,d(d^2-\Delta^2)}{4\,Gm^2\,\Delta^2} \\ &=\; \frac{\pi}{2}\,\frac{d(d^2-\Delta^2)}{\Delta^2}\Big[\frac{6}{5R} - \frac{1}{\Delta}\Big] \;=\; \frac{\pi}{2}\,\frac{d^3}{\Delta^2}(1-(\Delta/d)^2)\Big[\frac{6}{5R} - \frac{1}{\Delta}\Big]\,, \end{aligned} \tag{59}$$

which is Eq. (25). In the deep-saturated regime $\Delta \gg 2R$ the bracket $\to 6/(5R)$ and this reduces to the square law $(3\pi/5)(d^3/R\Delta^2)(1-(\Delta/d)^2)$ of Eq. (26); the sub-leading $-(\Delta/d)^2$ phase factor carries a positive sign in the visibility (less suppression than the leading-order formula), because the exact phase reaches the witness threshold faster than its leading-order approximation predicts.

A.2 Perpendicular-arm geometry: from energy to visibility

The four-branch interaction energies for the perpendicular-arm layout are

$$E_{LL} = E_{RR} = -\frac{Gm^2}{d}, \qquad E_{LR} = E_{RL} = -\frac{Gm^2}{\sqrt{d^2+\Delta^2}}. \tag{60}$$

The witness phase difference is

$$|\Delta\phi_{\rm perp}(\tau)| \;=\; \frac{2\,Gm^2\,\tau}{\hbar}\left[\frac{1}{d} - \frac{1}{\sqrt{d^2+\Delta^2}}\right] \;=\; \frac{2\,Gm^2\,\tau}{\hbar\,d}\left[1 - \frac{1}{\sqrt{1+u^2}}\right]. \tag{61}$$

Setting this equal to $\pi/2$ gives

$$\tau_{\rm BMV}^{\rm perp,exact} \;=\; \frac{\pi\hbar\,d}{4Gm^2}\,\frac{1}{1-1/\sqrt{1+u^2}} \;=\; \frac{\pi\hbar\,d}{4Gm^2}\,\frac{\sqrt{1+u^2}}{\sqrt{1+u^2}-1}. \tag{62}$$

Rationalising the denominator,

$$\frac{\sqrt{1+u^2}}{\sqrt{1+u^2}-1} \;=\; \frac{\sqrt{1+u^2}\,(\sqrt{1+u^2}+1)}{(1+u^2)-1} \;=\; \frac{1+u^2+\sqrt{1+u^2}}{u^2}. \tag{63}$$

Hence

$$\tau_{\rm BMV}^{\rm perp,exact} \;=\; \frac{\pi\hbar\,d}{4Gm^2}\,\frac{1+u^2+\sqrt{1+u^2}}{u^2}. \tag{64}$$

The two-particle visibility exponent, with the saturated rate, is then

$$\begin{aligned} -\ln V_{\rm exact}^{\rm perp} \;&=\; 2\Gamma_{\rm self}(\Delta)\,\tau_{\rm BMV}^{\rm perp,exact} \;=\; 2\,\frac{Gm^2}{\hbar}\left[\frac{6}{5R} - \frac{1}{\Delta}\right]\cdot\frac{\pi\hbar\,d}{4Gm^2}\,\frac{1+u^2+\sqrt{1+u^2}}{u^2} \\ &=\; \frac{\pi}{2}\,d\,\frac{1+u^2+\sqrt{1+u^2}}{u^2}\left[\frac{6}{5R} - \frac{1}{\Delta}\right], \end{aligned} \tag{65}$$

which is Eq. (29). In the deep-saturated wide-arm limit ($\Delta \gg 2R$, then $u \to 0$), the bracket $\to 6/(5R)$ and $1+u^2+\sqrt{1+u^2} \to 2$, so $-\ln V_{\rm exact}^{\rm perp} \to (6\pi/5)\,d^3/(R\Delta^2)$, recovering the leading square law of Eq. (21).

A.3 Sub-leading expansion

Expanding both exact-phase exponents in u in the deep-saturated regime (bracket $\to 6/(5R)$):

$$\begin{aligned} -\ln V_{\rm exact}^{\rm par} \;&=\; \frac{3\pi}{5}\,\frac{d}{R}\,(u^{-2}-1) \;=\; \frac{3\pi}{5}\,\frac{d}{R}\,u^{-2} - \frac{3\pi}{5}\,\frac{d}{R}, \\ -\ln V_{\rm exact}^{\rm perp} \;&=\; \frac{3\pi}{5}\,\frac{d}{R}\,\frac{1+u^2+\sqrt{1+u^2}}{u^2} \;=\; \frac{6\pi}{5}\,\frac{d}{R}\,u^{-2} + \frac{9\pi}{10}\,\frac{d}{R} + \mathcal{O}(u^2). \end{aligned} \tag{66}$$

The leading-order coefficients are $3\pi/5$ (parallel) and $6\pi/5$ (perpendicular), giving the factor-of-two prefactor difference of Eq. (22). The u-independent sub-leading terms are $-(3\pi/5)(d/R)$ (parallel) and $+(9\pi/10)(d/R)$ (perpendicular), with opposite signs: the parallel-axis exact formula is more permissive than its leading-order limit by $\exp(+(3\pi/5)(d/R))$, consistent with the ratio Eq. (27); the perpendicular-axis exact formula is more suppressive by $\exp(-(9\pi/10)(d/R))$.

B Algebra of the Oblique-Arm and Magic-Angle Formulae

The algebraic steps from the angular branch distances of Eq. (34) to the multipole expansion of Eq. (36), and thence to the exact magic-angle series (44), with $u \equiv \Delta/d$ and displacement vector $\hat{n} = (\cos\theta, \sin\theta, 0)$.

B.1 Legendre generating function for the branch distances

The two non-trivial branch distances at general θ are $r_\pm(\theta) = d\sqrt{1 \pm 2u\cos\theta + u^2}$. Their reciprocals admit the Legendre generating function expansion

$$\frac{1}{r_\pm(\theta)} = \frac{1}{d}\sum_{\ell\geq 0}(\mp 1)^\ell\, P_\ell(\cos\theta)\, u^\ell\,, \tag{67}$$

valid for $u < 1$. Summing the two branches, the odd-ℓ terms cancel,

$$\frac{1}{r_+(\theta)} + \frac{1}{r_-(\theta)} = \frac{2}{d}\sum_{\ell\geq 0} P_{2\ell}(\cos\theta)\, u^{2\ell}\,. \tag{68}$$

The witness phase combination of Eq. (35) is therefore

$$\frac{2}{d} - \frac{1}{r_+} - \frac{1}{r_-} = -\frac{2}{d}\sum_{\ell\geq 1} P_{2\ell}(\cos\theta)\, u^{2\ell}\,. \tag{69}$$

Inserting the standard Legendre values

$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)\,, \qquad P_4(\cos\theta) = \tfrac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)\,, \tag{70}$$

and taking the modulus gives, with $|\Delta\phi| = (Gm^2\tau/\hbar)\big|\,2/d - 1/r_+ - 1/r_-\,\big|$,

$$\begin{aligned} |\Delta\phi(\theta)| &= \frac{Gm^2\tau}{\hbar d}\Big[\,2\,|P_2|\,u^2 + 2\,|P_4|\,u^4 + \mathcal{O}(u^6)\Big] \\ &= \frac{Gm^2\tau}{\hbar d}\Big[\,|1 - 3\cos^2\theta|\,u^2 + \tfrac{1}{4}|3 - 30\cos^2\theta + 35\cos^4\theta|\,u^4 + \mathcal{O}(u^6)\Big]\,, \end{aligned} \tag{71}$$

recovering Eq. (36). The leading u^2 angular factor is exactly $-2\,P_2(\cos\theta)$ in modulus; its zero $\cos^2\theta_m = 1/3$ defines the magic angle of Eq. (37).

B.2 Magic-angle phase function $g_m(u)$

At $\theta = \theta_m$, $\cos\theta_m = 1/\sqrt{3}$, so $2\cos\theta_m = 2/\sqrt{3}$. Define

$$x_\pm(u) \equiv \left(1 \pm \tfrac{2}{\sqrt{3}}u + u^2\right)^{-1/2} = \sum_{\ell\geq 0}(\mp 1)^\ell\, P_\ell(1/\sqrt{3})\, u^\ell\,, \tag{72}$$

so that $g_m(u) = |2 - x_+(u) - x_-(u)|$ as in Eq. (43). From the parity-summed series,

$$g_m(u) = \Big|-2\sum_{\ell\geq 1} P_{2\ell}(1/\sqrt{3})\, u^{2\ell}\Big|\,. \tag{73}$$

The relevant Legendre values are

$$
\begin{aligned}
P_2(1/\sqrt{3}) &= \tfrac{1}{2}\,(3\cdot\tfrac{1}{3}-1) \;=\; 0\,,\\
P_4(1/\sqrt{3}) &= \tfrac{1}{8}\left(\tfrac{35}{9}-\tfrac{30}{3}+3\right) \;=\; \tfrac{1}{8}\cdot\tfrac{35-90+27}{9} \;=\; -\tfrac{7}{18}\,,\\
P_6(1/\sqrt{3}) &= \tfrac{1}{16}\left(\tfrac{231}{27}-\tfrac{315}{9}+\tfrac{105}{3}-5\right) \;=\; \tfrac{1}{16}\cdot\tfrac{231-945+945-135}{27} \;=\; \tfrac{2}{9}\,.
\end{aligned}
\tag{74}
$$

The u^2 coefficient of g_m vanishes (this is the magic-angle defining property $P_2(1/\sqrt{3}) = 0$); the u^4 coefficient is $-2P_4(1/\sqrt{3}) = 7/9$; the u^6 coefficient is $-2P_6(1/\sqrt{3}) = -4/9$. Hence

$$
g_m(u) \;=\; \tfrac{7}{9}\,u^4 \;-\; \tfrac{4}{9}\,u^6 \;+\; \mathcal{O}(u^8) \;=\; \tfrac{7}{9}\,u^4\left[1-\tfrac{4}{7}\,u^2+\mathcal{O}(u^4)\right], \tag{75}
$$

which is Eq. (44). The leading coefficient is exactly 7/9, reproducing Eq. (39); combined with the saturated self-energy this gives the deep-saturated quartic $V_{\rm lead}^{\rm magic} = \exp[-(54\pi/35)(d/\Delta)^4(d/R)]$ of Eq. (41) (the bare phase series alone, paired with the old point rate Gm^2/Δ, would have given the point-mass quintic $\exp[-(9\pi/7)(d/\Delta)^5]$, which survives only at the $\Delta \to 2R^+$ edge). The first phase correction enters at relative order u^2 with negative coefficient $-4/7$.

The choice of absolute value in Eq. (43) requires the two-term series to be sign-checked for the range $u \in [0,1]$. Equation (75) is positive in this range ($1-(4/7)u^2 > 1-4/7 = 3/7 > 0$), and direct numerical evaluation of $|2 - x_+(u) - x_-(u)|$ confirms positivity at the few percent level for all $u \in [0,1]$ checked. Numerically at $u = 0.5$: the two-term truncation gives $g_m \approx (7/9)(0.0625)(1-1/7) = 0.0417$; the exact closed form Eq. (43) gives $g_m(0.5) = 0.04097$, agreeing to better than 2×10^{-2} relative accuracy. At $u = 0.1$ the agreement is sub-permille; at $u = 0.01$ the relative accuracy is sub-ppm ($\sim 10^{-7}$) — the series is the appropriate evaluation below the seam $u_{\rm seam} = 10^{-2}$ where the closed form Eq. (43) loses thirteen significant digits to catastrophic cancellation ($x_+ + x_- \to 2$).

B.3 Exact-phase visibility expansion

The exact-phase magic-angle visibility uses the *unchanged* witness phase $g_m(u)$ above, with the finite-size saturated self-decoherence rate $\Gamma_{\rm self}(\Delta) = (Gm^2/\hbar)[6/(5R) - 1/\Delta]$ of Eq. (12) (Section 2). Setting $|\Delta\phi(\theta_m)| = (Gm^2\tau/\hbar d)g_m(u) = \pi/2$ gives $\tau_{\rm BMV}^{\rm magic,exact} = (\pi\hbar d/2Gm^2)/g_m(u)$, and the two-particle exponent $2\Gamma_{\rm self}\tau$ (with Gm^2 cancelling) is

$$
-\ln V_{\rm exact}^{\rm magic} \;=\; \frac{\pi d}{g_m(u)}\left[\frac{6}{5R}-\frac{1}{\Delta}\right]. \tag{76}
$$

In the deep-saturated regime $\Delta \gg 2R$ the bracket $\to 6/(5R)$; substituting Eq. (75),

$$
-\ln V_{\rm exact}^{\rm magic} \xrightarrow{\Delta\gg 2R} \frac{6\pi d}{5R}\cdot\frac{9}{7u^4}\left[1-\tfrac{4}{7}u^2\right]^{-1} \;=\; \frac{54\pi}{35}\,\frac{d^5}{R\Delta^4}+\frac{216\pi}{245}\,\frac{d^3}{R\Delta^2}+\mathcal{O}(d/R)\,, \tag{77}
$$

which is Eq. (46). The sub-leading correction has the *positive* sign ($+216\pi/245$) and therefore strengthens the suppression relative to the pure-quartic leading term, as stated in Eq. (47). The factor $\frac{54}{35} = \frac{9}{7}\cdot\frac{6}{5}$ and $\frac{216}{245} = \frac{36}{49}\cdot\frac{6}{5}$ inherit the old phase-series coefficients multiplied by the saturated energy factor 6/5.

B.4 Cardinal-geometry consistency check

The Legendre generating-function form (69) reproduces the parallel-axis and perpendicular results of Sections 3 and 4 at $\theta = 0$ and $\theta = \pi/2$ respectively. At $\theta = 0$, $\cos\theta = 1$, and $1/r_{\pm}(0) = (1/d)(1 \mp u)^{-1} = (1/d)\sum_{\ell}(\mp 1)^{\ell} u^{\ell}$, recovering Eq. (13). At $\theta = \pi/2$, $\cos\theta = 0$ and only the $u^{2\ell}$ terms with $P_{2\ell}(0)$ non-zero contribute. Eq. (67) gives $\sum_{\ell} P_{2\ell}(0) u^{2\ell} = (1+u^2)^{-1/2}$ (the standard Taylor series at $x = 0$), so

$$\frac{1}{r_+(\pi/2)} + \frac{1}{r_-(\pi/2)} \;=\; \frac{2}{d}\,(1+u^2)^{-1/2}\,, \tag{78}$$

recovering Eq. (18). Both cardinal geometries are therefore reproduced from the same Legendre identity, with the magic angle representing the unique zero of the leading multipole.

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