Geometric Modular Flow for Gravitons in a Causal Diamond:

Dimensional Reduction and the Type II Crossed Product for a Non-Conformal Field

Quantum-Geometric Correspondence

Marc Sperzel*

Independent Researcher

MSci Physics, King's College London

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Abstract

The modular Hamiltonian of a causal diamond in the Minkowski vacuum is geometric—local, with the conformal-Killing weight $f(r) = (R^2 - r^2)/(2R)$—for conformally invariant fields, a fact underlying recent Type II crossed-product constructions in which the diamond's modular flow is an observer's proper time. The linearized graviton is not conformally invariant, and the conformal map to $\mathbb{R} \times \mathbb{H}^3$ behind these results does not act on it; this has been regarded as a structural obstruction to extending diamond modular theory to the physical spin-2 field. We show the obstruction dissolves: the vacuum modular flow of the gauge-invariant graviton ball algebra is geometric. Three published results assemble: the graviton's gauge-invariant sphere content is two towers of massless-scalar spherical modes with $\ell \geq 2$ (Benedetti–Casini); rotational invariance makes the reduced quasi-free vacuum block-diagonal over the towers; and each dimensionally reduced, manifestly non-conformal angular mode inherits a local modular Hamiltonian with the parent weight $f(r)$ that generates its flow (Huerta–van der Velde). The mode equivalence has an elementary anchor: at $M \to 0$ the Regge–Wheeler and Zerilli master potentials degenerate to the scalar's $\ell(\ell+1)/r^2$, verified symbolically for general ℓ and on a radial lattice at machine precision. The assembly is validated on the Maxwell field, where it reproduces the answer known independently from conformal invariance; the helicity ladder two scalars $\supset$ Maxwell ($\ell \geq 1$) $\supset$ graviton ($\ell \geq 2$) is exact mode by mode. Consequences: the diamond's Tolman temperature is helicity-blind; the graviton crossed product is Type II_∞ with a canonical trace (its anomaly-dependent normalization left open); and a centrally localized excitation of energy ΔE accumulates modular phase at the proper-time rate $\Delta E/\hbar$, independent of the diamond radius. The remaining open items—edge modes, the trace normalization, the intrinsic observer, and the reduction hypothesis for spins $s \geq 3$—are stated explicitly.

*ORCID: 0009-0000-6252-3155. Email: me@marcsperzel.com

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1 Introduction

The modular Hamiltonian of a region—the logarithm of the reduced density matrix of the vacuum—is, for most regions and most theories, a non-local operator with no closed form. The known exceptions are precious. For the Rindler wedge of any Wightman theory, the Bisognano–Wichmann theorem identifies the modular Hamiltonian with 2π times the boost generator [1]. For a causal diamond, locality of the modular Hamiltonian is known only for conformally invariant fields: Hislop and Longo established the geometric modular action for the massless scalar [2], and Casini, Huerta and Myers exhibited the generator explicitly [3],

$$K \;=\; 2\pi \int_{r<R} d^3x\, f(r)\, T_{00}(x), \qquad f(r) = \frac{R^2 - r^2}{2R}, \tag{1}$$

via the conformal map that takes the diamond to $\mathbb{R} \times \mathbb{H}^3$ and its modular flow to a genuine Killing time-translation.

The geometric character of (1) has acquired a structural role in quantum gravity. The crossed-product constructions of Type II von Neumann algebras—initiated for the de Sitter static patch by Chandrasekaran, Longo, Penington and Witten [23], following [22], and extended to general subregions by Jensen, Sorce and Speranza [24]—identify the modular flow of a causal region with the proper time of an observer who carries a clock along it. The identification is physically clean precisely when the flow is geometric: a flow that acts as a time-translation is one an observer can follow. For a finite diamond and conformal matter, the Casini–Huerta–Myers (CHM) map supplies this geometric structure exactly.

The graviton breaks the pattern. The Fierz–Pauli action [15] is not conformally invariant—among free massless gauge fields in four dimensions, the Maxwell field and the conformally coupled scalar are conformal field theories, the graviton is not—so the CHM map does not act on it, and the route that establishes (1) is unavailable. Since the gravitational field is the one field a theory of quantum gravity cannot treat as optional, this looks like a structural obstruction: diamond modular theory, and the Type II constructions built on it, would apply to every field *except* the one that matters most. The prevailing expectation, shaped by the failure of the conformal route at its very first step, has accordingly been that the method does not extend to the physical spin-2 field.

This paper shows that the expectation conflates a sufficient condition with a necessary one. Conformal invariance of the action is one way to establish geometric modular flow; it is not the only way. The modular flow of a region is intrinsic to the pair (algebra, state). The right question is therefore not "does the CHM map act on the graviton action?" but

> *Does the gauge-invariant graviton algebra of a ball, in the Minkowski vacuum, have geometric modular flow?*

We show that the answer is yes, and that every load-bearing ingredient is already in the published literature; what has been missing is their assembly.

1.1 The argument in brief

Three results combine.

1. **Dimensional reduction of the graviton ball algebra.** Benedetti and Casini [4] decomposed the linearized graviton in tensor spherical harmonics on the $t = 0$ ball and fixed the gauge so that the two dynamical modes for each angular momentum (ℓ, m) decouple, each with the dynamics of a massless-scalar spherical mode; the $\ell = 0, 1$ towers are absent (they carry no radiative degrees of freedom), and the gauge-fixed degrees of freedom inside the sphere represent gauge-invariant operators localized in the same region. The graviton ball algebra is two scalar towers with $\ell \geq 2$.

2. **Block-diagonality.** The Minkowski vacuum is Gaussian and rotation invariant, so its one-particle data restricted to the ball—the symplectic form together with the correlators—are block-diagonal over (ℓ, m, parity), and by Araki's modular theory of quasi-free states [17, 18] so is the modular operator. The modular flow does not mix the towers (Section 3.3).

3. **Per-mode geometric flow without conformal invariance.** The dimensionally reduced angular modes are half-line theories with a centrifugal potential; they are *not* conformally invariant. Nevertheless, Huerta and van der Velde [5] showed that for an interval $(0, R)$ attached to the origin, the modular Hamiltonian of each reduced mode is local in the energy density with the parent weight $f(r)$ of (1), and that this local operator generates the modular flow—the surviving portion of the parent conformal symmetry suffices. Geometric modular flow, one non-conformal tower at a time.

Chaining the three: the vacuum modular flow of the gauge-invariant graviton ball algebra is the CHM conformal-Killing flow restricted to the $\ell \geq 2$ double tower, with generator

$$K_{\text{grav}} \;=\; 2\pi \sum_{P=\pm} \sum_{\ell \geq 2} \sum_{|m| \leq \ell} \int_0^R dr\; f(r)\; T_{00}^{(\ell,m,P)}(r), \tag{2}$$

where $T_{00}^{(\ell,m,P)}$ is the energy density of the reduced one-dimensional mode of parity P, Eq. (6). The graviton's diamond modular flow is geometric—for a field that is not a conformal field theory.

The mode equivalence in step 1 has an independent, elementary anchor: it is the flat-space degeneration of black-hole perturbation theory. The odd-parity (Regge–Wheeler [10]) and even-parity (Zerilli [11]) master potentials on a Schwarzschild background of mass M are distinct from each other and from the massless scalar's potential at $M \neq 0$; at $M = 0$ all three collapse to the same centrifugal potential $\ell(\ell + 1)/r^2$, and the common master solutions are the Riccati–Bessel functions $r\, j_\ell(\omega r)$—the scalar spherical modes. We verify this degeneration symbolically for general ℓ and at machine precision on a radial lattice (Section 6), where the per-mode equivalence is confirmed at the level of symplectic spectra, trace invariants, and entanglement entropies.

The assembly is validated on a case where the answer is independently known: the Maxwell field, whose gauge-invariant sphere content is two scalar towers with $\ell \geq 1$ [6], is a CFT, so its geometric diamond flow follows from *both* the conformal route and the reduction route. The two agree (Proposition 3.3), and the graviton then differs from Maxwell only in where its towers start. The resulting helicity ladder—two scalars $\supset$ Maxwell ($\ell \geq 1$) $\supset$ graviton ($\ell \geq 2$)—is verified mode by mode on the lattice.

1.2 What follows from it, and what does not

The geometric flow carries three consequences (Section 5). The local (Tolman) temperature of the diamond, $T_{\rm loc}(r) = 1/(2\pi f(r))$, applies to the graviton towers as to conformal matter: the diamond's thermality is helicity-blind. The crossed product of the graviton diamond algebra by its modular flow is a Type II_∞ von Neumann algebra with a canonical trace [21, 22, 23], and the geometric character of the flow places the physical observer-clock identification of [23, 24] on the same footing for the graviton as for conformal matter. And any centrally localized excitation of energy ΔE accumulates modular phase at the proper-time rate $\Delta E/\hbar$, independently of the arbitrary diamond radius R—a statement that now holds uniformly across helicity, and that bears on proposed gravitational-decoherence mechanisms in which a modular clock sets the rate (Section 5.3).

Equally important is what this paper does *not* claim (Section 7). The result concerns the modular *flow*; the entropy bookkeeping of the graviton diamond is subtler. The graviton is not globally equivalent to two conformal scalars—the universal logarithmic coefficient of its sphere entropy is $-61/45$ [4], far from twice the scalar's $-1/90$ [27]—because the missing $\ell = 0, 1$ towers and the algebra's edge structure contribute; edge modes and the center of the algebra affect entropy constants and are the subject of an active literature [7, 8, 9], but not the bulk flow. The observer clock of the crossed product remains *transcribed* from the de Sitter construction rather than built intrinsically for the diamond—a residue our result renders no longer graviton-specific at the flow level, though graviton-specific subtleties of the transcription itself (constraint structure, edge sector) remain possible. The trace-anomaly normalization of the crossed-product trace is not computed here. And (2) is a mode-sum statement on the gauge-invariant algebra: we do not claim a covariant local gauge-invariant energy density for the graviton, which does not exist in the standard sense. All results are at linearized order in the gravitational field.

1.3 Relation to prior work

Every ingredient we use is published: [4] for the reduction, [5] for the per-mode modular Hamiltonian, [2, 3] for the parent geometric flow, [21, 22, 23, 24] for the crossed product, [6] for the Maxwell reduction. The contribution of this paper is the assembly and its consequence: to our knowledge, the statement that the *physical graviton's* diamond modular flow is geometric—and that the Type II construction therefore extends to it—has not been drawn before, and the prevailing expectation, shaped by the failure of the conformal route, has been the opposite. Corroborating evidence that the hyperbolic-cylinder frame captures graviton bulk physics despite non-conformality appears in the work of David and Mukherjee [7], who found that the Benedetti–Casini sphere entropy is reproduced by a Kaluza–Klein tower on $S^1 \times AdS_3$, with edge modes accounting for the difference in a companion analysis [8]. The flat-space degeneration of the Regge–Wheeler and Zerilli equations is classical [10, 11, 14]; its role as a first-principles anchor for diamond modular theory appears to be new.

Section 2 fixes conventions and states the problem. Section 3 assembles the main result and the Maxwell validation. Section 4 constructs the crossed product. Section 5 develops the consequences, Section 6 the numerical and symbolic verification, and Section 7 the honest residues and the general spin-s outlook.

2 Setup: diamond modular flow and the proxy question

2.1 Modular theory of the diamond

Let D be the causal diamond over the $t = 0$ ball $B_R = \{r < R\}$ in Minkowski space, $\mathcal{A}(D)$ the von Neumann algebra of observables localized in D, and Ω the Minkowski vacuum. Tomita–Takesaki theory assigns to the pair $(\mathcal{A}(D), \Omega)$ a modular operator Δ and the one-parameter modular automorphism group $\sigma_s = \mathrm{Ad}\,\Delta^{is}$; the modular Hamiltonian is $K = -\ln\Delta$. For a general region and theory, σ_s acts non-locally. The flow is called *geometric* when it acts by point transformations—when there is a vector field ξ on D such that σ_s moves operators along the flow of ξ.

For conformally invariant fields the diamond flow is geometric [2]: ξ is the conformal Killing vector preserving D,

$$\xi \;=\; \frac{\pi}{R}\Big[\,\tfrac{1}{2}(R^2 - t^2 - r^2)\,\partial_t \;-\; t\,r\,\partial_r\Big], \tag{3}$$

normalized so that $\xi^t|_{t=0} = \pi f(r)$ with $f(r) = (R^2 - r^2)/(2R)$; the generator on the $t = 0$ slice is the CHM expression (1) [3], whose flow has lapse $2\pi f(r)$, so σ_s is the flow of (3) at parameter $2s$. Two features of this structure matter below; we work in units $\hbar = c = k_B = 1$ except where stated.

First, the weight $f(r)$ vanishes at the edge $r = R$ and peaks at the center, $f(0) = R/2$; the associated local temperature

$$T_{\rm loc}(r) \;=\; \frac{1}{2\pi f(r)} \tag{4}$$

is the temperature registered by a static detector that follows the modular flow at radius r.

Second, the route to (1) runs through conformal symmetry, and its geometry is explicit [3]: the Weyl rescaling $\hat{g} = \Omega^{-2}\eta$ with

$$\Omega^2 \;=\; \frac{[(R+t)^2 - r^2]\,[(R-t)^2 - r^2]}{R^4} \tag{5}$$

maps D to the static cylinder $\mathbb{R}\times\mathbb{H}^3$; the $t = 0$ ball becomes the hyperbolic space $\mathbb{H}^3$ of curvature radius $R/2$, with the diamond boundary $r = R$ sent to its infinity; and ξ becomes an exact Killing vector of constant norm, $|\xi|^2_{\hat{g}} = -\pi^2 R^2/4$—a globally static time-translation with no interior horizon. The Tolman relation $T_{\rm loc}(r)\,\Omega(r) = \text{const}$ exhibits the diamond temperature profile (4) as the blueshifted image of the uniform cylinder temperature. For a conformal field the algebra and vacuum transport along this map, and the diamond inherits the cylinder's thermal structure.

2.2 Why the graviton was thought to be excluded

The transport in the last step requires conformal covariance of the field. The linearized graviton does not have it: the Fierz–Pauli action [15] is not Weyl invariant, the graviton is not a CFT, and the CHM map simply does not act on the theory. Every step of the chain "conformal invariance ⇒ cylinder frame ⇒ Killing flow ⇒ geometric modular Hamiltonian" fails at the first link.

If geometric modular flow *required* conformal invariance, that would be the end of the story, and the modular theory of the diamond would not apply to the gravitational field. The Type II crossed-product constructions [23, 24], which need the flow to be a time-translation an observer can follow, would inherit the same restriction. It is this expectation—natural, given the failure of the conformal route at its first step—that the present paper revisits.

The key observation is that the implication only runs one way. The modular flow of $(\mathcal{A}, \Omega)$ is determined by the algebra and the state; conformal invariance is a tool for *computing* it, not a precondition for its geometric character. There is a second route to the diamond modular Hamiltonian that never invokes conformal invariance of the full theory: dimensional reduction.

2.3 Dimensional reduction of the scalar diamond

A massless scalar in four dimensions, decomposed in spherical harmonics, is an infinite tower of half-line theories: for each (ℓ, m) the rescaled radial field $\psi_{\ell m}(r) = r\,\phi_{\ell m}(r)$, regular at the origin ($\psi \sim r^{\ell+1}$), has Hamiltonian $H_\ell = \int_0^\infty dr\, T_{00}^{(\ell)}(r)$ with reduced energy density

$$T_{00}^{(\ell)}(r) \;=\; \frac{1}{2}\Big[\pi_{\ell m}^2 + (\partial_r \psi_{\ell m})^2 + \frac{\ell(\ell+1)}{r^2}\,\psi_{\ell m}^2\Big]. \tag{6}$$

This explicit density is the object whose locality the whole construction turns on. The centrifugal potential breaks one-dimensional conformal invariance for every $\ell \geq 1$: the reduced theories are *not* CFTs. Nevertheless, the ball algebra of the parent scalar is the direct sum of the interval algebras of the modes, the vacuum reduces mode by mode, and the parent modular Hamiltonian (1) therefore decomposes into mode contributions. Huerta and van der Velde [5] analyzed the reduced theories directly and showed that for the interval $(0, R)$ attached to the origin each mode's modular Hamiltonian is local,

$$K_\ell \;=\; 2\pi \int_0^R dr\, f(r)\, T_{00}^{(\ell)}(r), \tag{7}$$

with the *same* weight $f(r)$ as the parent, and—the nontrivial part—that (7) generates the modular flow of the reduced mode. The symmetries that survive the reduction (the subgroup of the parent conformal group commuting with the angular decomposition) are enough; full conformal invariance of the reduced theory is not needed.

This is the mechanism we exploit for the graviton: if the gauge-invariant graviton ball algebra decomposes into towers whose reduced data—radial Hamiltonian, origin boundary condition, and vacuum—coincide with those of (6), then (7) applies tower by tower, and the graviton's diamond modular flow is geometric with no appeal to conformal invariance of the graviton action. The next section establishes exactly this decomposition.

3 The graviton ball algebra and its modular flow

3.1 The Benedetti–Casini reduction

Benedetti and Casini [4] analyzed the linearized graviton $h_{\mu\nu}$ on the $t = 0$ ball: decompose $h_{\mu\nu}$ in tensor spherical harmonics (Regge–Wheeler conventions [10, 12]) and fix the gauge so that, for each angular momentum (ℓ, m), the two dynamical (radiative) modes decouple, one per parity sector. Their results, in the form we need:

(R1) For each (ℓ, m) with $\ell \geq 2$ there are exactly two dynamical modes (one even-parity, one odd-parity), and each has the dynamics of a massless-scalar spherical mode: the reduced radial Hamiltonians are those of the dimensionally reduced scalar, Eq. (6), at the same ℓ, with the same regularity condition $\psi \sim r^{\ell+1}$ at the origin.

(R2) The $\ell = 0$ and $\ell = 1$ sectors contain no radiative modes. This is the angular-momentum form of the Arnowitt–Deser–Misner (ADM) count [16]: of the ten components of $h_{\mu\nu}$, four are gauge and four are fixed by constraints, leaving the two helicities, which appear first at $\ell = 2$. The low-ℓ sectors are constraint/gauge data (mass, momentum, center of mass, angular momentum), not local degrees of freedom.

(R3) The gauge-fixed field degrees of freedom inside the sphere represent gauge-invariant operators of the theory localized in the same region: the reduction is a statement about the *local algebra*, not merely about mode functions.

(R4) Consequently the sphere entanglement entropy of the graviton equals that of a pair of massless scalars with the $\ell = 0, 1$ contributions removed, with universal logarithmic coefficient $-61/45$.

Of these, (R3) is the load-bearing statement for modular theory: it identifies the gauge-invariant graviton ball algebra as

$$\mathcal{A}_{\rm grav}(B_R) \;\cong\; \bigotimes_{P=\pm} \;\; \bigotimes_{\ell\geq 2,\,|m|\leq\ell} \mathcal{A}_\ell^{(m,P)}((0,R)), \tag{8}$$

where each factor is the interval algebra of a reduced scalar mode (6), and the Minkowski vacuum restricts to the corresponding mode vacuum factor by factor. (The algebra tensor-factorizes over the blocks; correspondingly, the one-particle structures introduced in Section 3.3 decompose as a direct sum.)

3.2 A first-principles anchor: the flat-space master equations

Statement (R1) can look like an accident of gauge fixing. It is not; it is the flat-space limit of a classical fact of black-hole perturbation theory, and exhibiting it this way makes the per-mode equivalence checkable by elementary means.

On a Schwarzschild background of mass M, the two parities of the graviton obey master equations $-\psi'' + V(r)\psi = \omega^2\psi$ (primes denoting tortoise-coordinate derivatives) with the Regge–Wheeler [10] and Zerilli [11] potentials

$$V_{\rm RW} = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right], \tag{9}$$

$$V_{\rm Z} = \left(1 - \frac{2M}{r}\right)\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 M r^2 + 18\lambda M^2 r + 18M^3}{r^3\,(\lambda r + 3M)^2}, \qquad \lambda = \tfrac{(\ell-1)(\ell+2)}{2}, \tag{10}$$

while a massless scalar obeys the same equation with $V_{\rm s} = (1 - 2M/r)\,[\ell(\ell+1)/r^2 + 2M/r^3]$. At $M \neq 0$ the three potentials are pairwise distinct (the relation between (9) and (10) is isospectrality via a Darboux transformation [13, 14], not equality). At $M = 0$ they degenerate:

$$V_{\rm RW}\big|_{M=0} \;=\; V_{\rm Z}\big|_{M=0} \;=\; V_{\rm s}\big|_{M=0} \;=\; \frac{\ell(\ell+1)}{r^2}, \tag{11}$$

where the Zerilli case uses the identity $(\ell-1)(\ell+2)+2=\ell(\ell+1)$; the tortoise coordinate degenerates to r; and the common master solutions are the Riccati–Bessel functions $\psi = r\, j_\ell(\omega r)$, regular at the origin ($\psi \sim r^{\ell+1}$)—precisely the scalar spherical modes of (6), with the same origin boundary condition. Both graviton parities, mode by mode in (ℓ, m), are scalar modes in flat space. We verify (11) symbolically for general ℓ, and the resulting lattice couplings agree to the last floating-point bit while being pairwise distinct at $M \neq 0$ (Section 6)—the degeneration is exact and the comparison is not vacuous.

3.3 Block-diagonality of the modular operator

Lemma 3.1 (Block-diagonality)**.** *Let ω be a quasi-free (Gaussian) state of a free bosonic field, invariant under spatial rotations and parity, and let the region algebra decompose as in (8) into (ℓ, m, P) blocks generated by variables localized in the region. Then the modular operator of $(\mathcal{A}(B_R), \omega)$ is the second quantization of a direct sum over the blocks: the modular flow acts within each tower.*

Argument. For a free Bose field, the local von Neumann algebra is generated by Weyl operators over the real symplectic space (S_A, σ_A) of initial data supported in B_R, and the quasi-free state is specified by a covariance on S_A—concretely, by the pair of restricted correlators (X_A, P_A) of Appendix B. By Araki's theory of quasi-free states and the Tomita–Takesaki structure of free-field algebras [17, 18], the modular operator of the pair (Weyl algebra of (S_A, σ_A), quasi-free state) is the second quantization of a one-particle modular operator built from (σ_A, X_A, P_A) alone. Now use the symmetry twice: the ball is rotation and parity invariant, so the restriction to B_R commutes with the $SO(3)\times$ parity action; and the symplectic form, the vacuum correlators, and hence the full one-particle data are invariant under that action. Both σ_A and (X_A, P_A) are therefore simultaneously block-diagonal over the (ℓ, m, P) decomposition, the one-particle modular operator decomposes as a direct sum of the blocks' modular operators, and its second quantization factorizes over the towers. The one hypothesis doing real work is that the block variables are themselves localized in the ball—which is exactly statement (R3) of the Benedetti–Casini reduction. □

3.4 The main result

Theorem 3.2 (Geometric modular flow for the graviton diamond)**.** *The vacuum modular flow of the gauge-invariant linearized-graviton algebra $\mathcal{A}_{\text{grav}}(B_R)$ of a ball B_R in Minkowski space is geometric: it acts within each (ℓ, m, P) block as the modular flow of the corresponding reduced scalar mode, which is generated by the local operator (7); equivalently, the full generator is the mode sum*

$$K_{\text{grav}} \;=\; 2\pi \sum_{P=\pm} \sum_{\ell \geq 2} \sum_{|m| \leq \ell} \int_0^R dr\, f(r)\, T_{00}^{(\ell,m,P)}(r), \qquad f(r) = \frac{R^2 - r^2}{2R}, \tag{12}$$

with $T_{00}^{(\ell,m,P)}$ the reduced energy density (6): the Hislop–Longo/CHM conformal-Killing flow restricted to the $\ell \geq 2$ double tower.

Assembly. By the Benedetti–Casini reduction (R1)–(R3), $\mathcal{A}_{\text{grav}}(B_R)$ decomposes as (8) with the vacuum reducing mode by mode to the scalar mode vacua; the flat master-equation degeneration (11) supplies the per-mode identity independently. By Lemma 3.1 the modular operator is block-diagonal over the towers. On each block, the (algebra, state) pair coincides with that of a reduced scalar mode on $(0, R)$—the same radial Hamiltonian (6), the same regularity condition $\psi \sim r^{\ell+1}$ at the origin, and the same vacuum—and this is precisely the setting of the Huerta–van der Velde theorem [5], which holds for every angular mode ℓ of the reduced scalar and hence in particular for $\ell \geq 2$: the modular Hamiltonian is the local expression (7) and generates the flow. Since the modular operator is intrinsic to the (algebra, state) pair, each graviton block inherits it. Summing blocks gives (12). □

Three remarks delimit the claim. First, the theorem is a statement at linearized order; interactions are outside its scope, as everywhere in free-field modular theory. Second, Eq. (12) is a mode sum over the gauge-invariant towers. We do *not* claim it can be rewritten as $2\pi \int d^3x\, f(r)\, T_{00}^{\mathrm{grav}}(x)$ for a covariant, gauge-invariant local energy density: no such density exists for the graviton in the standard sense, and the mode-sum form is what the algebra supports—and all that the applications below require. Third, the statement concerns the bulk tower algebra (8); the boundary (edge-mode/center) sector, which contributes to entropy constants, is discussed in Section 7.

3.5 Validation on a known case: the Maxwell field

The assembly can be tested against a case where the answer is independently known. The Maxwell field *is* a conformal field theory in four dimensions, so its diamond modular flow is geometric by the classic conformal route [2, 3]. But Maxwell also admits a Benedetti–Casini-type reduction: Casini and Huerta [6] showed that the gauge-invariant Maxwell content of a sphere is two towers of massless-scalar spherical modes with the $\ell = 0$ mode removed (towers $\ell \geq 1$; one per helicity), with sphere entropy equal to two scalars minus their $\ell = 0$ contributions and universal logarithmic coefficient $-16/45$.

Proposition 3.3 (Maxwell cross-check)**.** *Applied to the Maxwell field, the assembly of Theorem 3.2—reduction [6], Lemma 3.1, and the per-mode theorem [5]—yields geometric modular flow with generator* $2\pi \sum \int f\, T_{00}^{(\ell,m,P)}$ *over the* $\ell \geq 1$ *double tower. This agrees with the result of the conformal route, which is available for Maxwell because it is a CFT.*

The two routes are logically independent: the conformal route transports the wedge result through the CHM map and never decomposes in modes; the reduction route never invokes conformal invariance of the action. Their agreement on Maxwell is therefore a genuine validation of the reduction route on the one gauge field where both apply—and the graviton then differs from Maxwell only in where its towers start ($\ell \geq 2$ instead of $\ell \geq 1$, two absent low-ℓ sectors instead of one). The resulting helicity ladder

$$\text{two scalars } (\ell \geq 0) \;\supset\; \text{Maxwell } (\ell \geq 1) \;\supset\; \text{graviton } (\ell \geq 2) \tag{13}$$

is verified on the lattice at machine precision in Section 6: the per-mode data are identical along the ladder, and the successive entropy deficits are exactly the dropped towers ($2S_0$ from scalars to Maxwell, $6S_1$ from Maxwell to graviton, per mode multiplicities).

4 The Type II crossed product for the graviton diamond

4.1 Takesaki: the trace exists for any modular flow

The local algebra of a causal diamond in a QFT vacuum is a Type III_1 factor [19, 20]: it admits no trace, no density matrices, and no canonical normalization of entropy. The crossed-product construction repairs this. For a Type III_1 factor, the crossed product by its own modular automorphism group,

$$\widehat{\mathcal{A}} = \mathcal{A}_{\mathrm{grav}}(B_R) \rtimes_\sigma \mathbb{R}, \tag{14}$$

is a Type II_∞ von Neumann algebra, with a faithful semifinite trace constructed from the dual weight and the added generator; this is Takesaki duality [21] in the form now standard in the quantum-gravity literature [22, 23]. The step is geometry-blind: it requires only the modular flow σ, which every (algebra, state) pair possesses, and in particular it never needed the CHM map. The *existence* of a Type II algebra with a trace for the graviton diamond was therefore never in doubt.

4.2 CLPW: the physical identification needs the flow to be geometric

What the conformal route supplied—and what was thought to fail for the graviton—is the *physical* content layered on top of (14) by Chandrasekaran, Longo, Penington and Witten [23] and by Jensen, Sorce and Speranza [24]: the crossed-product generator is identified with the energy of a gravitating observer, the modular flow with the observer's proper time, and the renormalized trace with a finite von Neumann (generalized) entropy. This identification is clean precisely when the modular flow is a geometric time-translation that an observer can carry a clock along. For the de Sitter static patch the flow is the boost Killing field and the identification is immediate; for the flat diamond and conformal matter, the CHM map turns the conformal-Killing flow into the constant-norm Killing time of $\mathbb{R} \times \mathbb{H}^3$, and the construction transcribes.

Theorem 3.2 supplies exactly the missing input for the graviton: its diamond modular flow *is* geometric—the same conformal-Killing flow (3), acting tower by tower. The geometry of the flow (the Weyl rescaling (5) to $\mathbb{R} \times \mathbb{H}^3$, the constant-norm Killing character of ξ there, the Tolman relation between the diamond and cylinder temperatures) is a statement about the *flow*, not about the matter content, and applies unchanged. The observer-clock identification of [23, 24] therefore carries over to the graviton diamond with the same standing it has for conformal matter.

Two clarifications keep the bookkeeping honest. First, "the same standing" includes the same caveat: for finite diamonds the CLPW observer is *transcribed* from the de Sitter construction—its bounded-below Hamiltonian and clock are carried over by structural analogy with the Killing case—rather than constructed intrinsically. Our result removes the flow-level obstruction that was specific to the graviton; whether the transcription itself harbors further graviton-specific subtleties (the constraint structure of the low-ℓ sector, the edge modes) is left open. Second, the trace normalization (equivalently the additive constant of the generalized entropy) involves the trace anomaly of the field content; for the graviton this constant is tied to its anomaly coefficients and is not computed here.

4.3 Independent corroboration: the graviton on the hyperbolic cylinder

That the cylinder frame captures graviton bulk physics despite the absence of conformal invariance is corroborated independently. David and Mukherjee [7] computed the entanglement entropy of $D = 4$ gravitons across a sphere from partition functions on the hyperbolic cylinder—a Kaluza–Klein tower of transverse-traceless massive spin-2 fields on $S^1 \times AdS_3$—and found agreement with the direct Benedetti–Casini mode computation for the bulk contribution; the difference is carried by edge modes, analyzed in [8] (see also [9] for the Rindler edge sector). In our language: the graviton's tower algebra transports to the cylinder frame mode by mode, exactly as Theorem 3.2 asserts, while the edge sector—which does not affect the flow on the tower algebra—accounts for the entropy bookkeeping that the naive cylinder computation misses.

5 Consequences

5.1 The diamond's thermality is helicity-blind

Because each graviton tower carries the same reduced (algebra, state) pair as the corresponding scalar mode, the local temperature seen by the modular flow,

$$T_{\rm loc}(r) \;=\; \frac{1}{2\pi f(r)} \;=\; \frac{R}{\pi(R^2 - r^2)}, \qquad T_{\rm loc}(0) = \frac{1}{\pi R}, \tag{15}$$

applies verbatim to graviton excitations. A finite causal diamond is a thermal object in a precise modular sense, and that thermality does not distinguish helicity 0 from helicity 1 or 2: a graviton wavepacket near the center of a diamond of radius R experiences the same local temperature $1/(\pi R)$ as a scalar or photon excitation. The Tolman product $T_{\rm loc}(r)\,\Omega(r)$ is constant across the diamond, with Ω the Weyl factor (5) of the cylinder map—the diamond temperature profile is the blueshifted image of the uniform cylinder temperature, now for the physical spin-2 field as well.

5.2 A helicity-blind modular clock, and the IR-safe phase rate

Consider an excitation of energy ΔE localized near the center of the diamond, $r \ll R$, in either a matter tower or a graviton tower (units $\hbar = c = 1$ in the intermediate steps). Restricted to centrally localized energy, the generator (12) reads $K \approx 2\pi f(0)\, H_{\rm loc} = \pi R\, H_{\rm loc}$. Two consequences follow at once: the excitation shifts the (dimensionless) modular Hamiltonian by $\Delta K = \pi R\,\Delta E$; and conjugation by e^{-iKs} acts on centrally localized operators as a time translation with lapse $2\pi f(0)$, so proper time advances at $d\tau/ds = 2\pi f(0) = \pi R$ per unit modular parameter s. The physical phase rate is their ratio:

$$(\text{modular phase rate in proper time}) \;=\; \frac{\Delta K}{d\tau/ds} \;=\; \Delta E \;\xrightarrow{\text{restoring units}}\; \frac{\Delta E}{\hbar}, \tag{16}$$

independent of the arbitrary diamond radius R. Both the modular gap and the modular-to-proper-time conversion scale as πR; the observable rate does not. The diamond one draws around a physical system leaves no imprint on the local clock—an infrared-safety statement—and, by Theorem 3.2, the clock that ticks in (16) is the *same* clock for matter, photons, and gravitons. The modular time of a causal diamond is universal across helicity.

5.3 Application: the propagating sector of gravitational decoherence

A long-standing hypothesis, going back to Diósi and Penrose [28, 29], holds that a spatial superposition of a mass M with branch separation d loses coherence at a rate set by the gravitational self-energy of the difference between its branches, $\Gamma \sim E_{\rm G}/\hbar$ with $E_{\rm G} = GM^2/d$ for point-like branches; at the benchmark $M = 1\ \mu$g, $d = 1$ mm this gives $E_{\rm G}/\hbar \approx 6.3 \times 10^8\,{\rm s}^{-1}$, a decoherence time of 1.6 ns. One program pursuing a microphysical derivation of this rate [30] rests on the identification of diamond modular flow with physical time: the relative modular phase between branches, evaluated through Eq. (16) with $\Delta E = E_{\rm G}$, reproduces the Diósi–Penrose rate up to an $O(1)$ coefficient. We do not re-derive or evaluate that program here; we record what the present result contributes to it.

The contribution concerns the *propagating* graviton sector. In such a derivation the interaction energy $E_{\rm G}$ resides in the constrained (non-radiative, $\ell = 0, 1$) sector, fixed by the matter configuration, and for static sources the transverse-traceless towers remain in their vacuum. But the total diamond algebra contains the TT towers as well, and if their modular flow had been non-geometric, the identification "modular flow = physical time" for the *total* algebra—matter, constraint sector, and radiation—would have carried an unquantified defect. Theorem 3.2 closes this specific failure mode: the TT towers flow geometrically with the same weight $f(r)$ as everything else, so the phase rate (16) holds uniformly across the diamond's field content; and since for static sources both branches carry the same TT vacuum, we expect no additional modular gap from the radiative sector at leading order in the source velocities. (The latter statement is the static limit, not a theorem; time-dependent sources radiate and would contribute at higher order.)

The epistemic status deserves emphasis: nothing here establishes the modular-time identification itself, which remains a hypothesis at the full operator level within that program's own accounting [30]; the present result removes one specific, previously unbounded failure mode (a non-geometric radiative sector) and leaves the remaining open items—the intrinsic observer construction and the trace normalization—common to all field content.

5.4 What the towers do not give: the entropy deficit

It is worth displaying explicitly what the per-mode equivalence does *not* imply, because the distinction is exactly where edge physics lives. At any fixed lattice regularization (Section 6), the graviton partial entropy over the towers satisfies the finite-sum identity

$$S_{\rm grav}(n) \;=\; 2\sum_{\ell\geq 2}(2\ell+1)\, S_\ell^{\rm scalar}(n), \tag{17}$$

which is *not* twice the full scalar entropy: the deficit is the $\ell = 0, 1$ contribution, $2\,[\,S_0 + 3S_1\,] > 0$. The helicity ladder (13) organizes this bookkeeping: Maxwell drops only the $\ell = 0$ pair (deficit $2S_0$), the graviton additionally drops the $\ell = 1$ towers (further deficit $6S_1$). In the continuum these deficits, together with the boundary/edge sector, are why the universal logarithmic coefficients differ from the naive two-scalar value $2 \times (-1/90) = -1/45$ [27]: $-16/45$ for Maxwell [6] and $-61/45$ for the graviton [4]. The flow statement of Theorem 3.2 lives on the towers and is exact; the entropy bookkeeping of the full diamond involves, in addition, the absent low-ℓ (constraint) sectors and the edge modes, and is the subject of Section 7.

6 Symbolic and numerical verification

All quantitative claims in this paper are machine-verified: the master-potential degeneration by exact computer algebra (ten symbolic identities, residuals identically zero), and the per-mode equivalence and its consequences by a lattice computation comprising 42 numerical checks. The code is available from the author and will be provided as ancillary files.

6.1 Exact symbolic checks

With V_{RW}, V_{Z}, V_{s} as in Section 3.2, computer algebra verifies, for *symbolic* ℓ: (i) $V_{\mathrm{RW}}|_{M=0} = V_{\mathrm{Z}}|_{M=0} = V_{\mathrm{s}}|_{M=0} = \ell(\ell+1)/r^2$ exactly, the Zerilli case reducing via $(\ell-1)(\ell+2)+2 = \ell(\ell+1)$; (ii) the three potentials are pairwise distinct at a generic point with $M \neq 0$; (iii) the tortoise measure degenerates, $dr_*/dr \to 1$; and (iv) $\psi = r\, j_\ell(\omega r)$ solves the common flat master equation for $\ell = 2, 3, 4$. All ten identities hold as rational expressions, not merely to numerical tolerance.

6.2 Lattice verification of the per-mode equivalence

The radial lattice method of Srednicki [25] (reviewed in [26]) discretizes each reduced mode on N sites and computes the reduced state on the inner n sites from ground-state correlators; the entanglement data are the symplectic eigenvalues ν_k of the restricted covariance (Appendix B). We run two discretizations: Srednicki's, for contact with the literature, and a direct half-line discretization of the master equation, in which the scalar and the two graviton parities are built from their *own* potential formulas V_{s}, $V_{\mathrm{RW}}|_{M\to 0}$, $V_{\mathrm{Z}}|_{M\to 0}$—so that the per-mode comparison tests the degeneration (11), not a shared code path.

The verified statements:

- **Baseline.** The scalar tower reproduces Srednicki's area law: $S(n)/(n+\frac{1}{2})^2 = 0.2948$ and 0.2952 at $n = 10, 20$ ($N = 60$), consistent with the classic coefficient ≈ 0.295 [25]; all symplectic eigenvalues respect the Heisenberg bound $\nu_k \geq \frac{1}{2}$; per-mode entropies decrease monotonically with ℓ.

- **Per-mode identity (the crux).** For $\ell = 2, \ldots, 6$ and both parities, the lattice couplings built from the graviton master potentials at $M \to 0$ agree with the scalar's to the last floating-point bit ($\max |\Delta K| = 0$); the trace invariants $\mathrm{Tr}\,[(X_A P_A)^k]$, $k = 1, \ldots, 4$, agree to better than 10^{-15} relative; sorted symplectic spectra and per-mode entanglement entropies agree to better than 10^{-12}.

- **Non-vacuousness.** At $M \neq 0$ (lattice units $M = 0.1$) the Regge–Wheeler, Zerilli, and scalar couplings are pairwise distinct at the 10^{-2} level: the $M \to 0$ degeneration is a real limit of genuinely different theories, not a definitional identity.

- **Mode bookkeeping.** Two helicities per (ℓ, m) for $\ell \geq 2$ and none for $\ell = 0, 1$: the tower count obeys $\sum_{\ell=2}^{L} 2(2\ell+1) = 2[(L+1)^2 - 4]$ (e.g. 234 at $L = 10$), the ADM count $10 - 4 - 4 = 2$, and the two-full-scalars-minus-eight-modes identity.

- **Tower sums and the deficit.** The finite-sum identity (17) holds to 10^{-12} relative with each side computed through its own potential formulas; the deficit from two full scalar towers equals the $\ell = 0, 1$ contribution and is positive; the per-mode identity is stable across lattice sizes $N = 40, 60, 100$.

- **The helicity ladder (Maxwell benchmark).** The mode counts obey $\sum_{\ell=1}^{L} 2(2\ell+1) = 2[(L+1)^2 - 1]$ for Maxwell, with ladder deficits of exactly 2 modes (the $\ell = 0$ pair, scalars $\to$ Maxwell) and 6 modes (the $\ell = 1$ towers, Maxwell $\to$ graviton) per $\ell_{\max}$; the entropy ladder $S_{2\,\text{scalars}} > S_{\text{Maxwell}} > S_{\text{grav}}$ holds with the successive differences equal to $2S_0$ and $6S_1$ as finite-sum identities at 10^{-12} relative—the lattice form of the nesting (13).

6.3 What the lattice does and does not test

The lattice locks the algebraic content of Theorem 3.2 and Proposition 3.3: the per-mode (algebra, state) equivalence, in the strong form of bitwise-equal couplings and machine-precision spectral data, plus the mode bookkeeping and tower sums along the helicity ladder. The geometricity of each tower's modular flow—the statement that (7) generates it—is the Huerta–van der Velde theorem [5], an analytic result we apply, not re-prove; direct lattice reconstruction of the modular kernel is known to converge slowly and is not used as evidence here. The division of labor is deliberate: machine precision where the new content lies (the equivalence), published proof where the literature already carries the load (the flow).

7 Discussion: scope, residues, and the general spin-s picture

7.1 What is established

At linearized order, on the gauge-invariant tower algebra of a ball: the graviton's vacuum modular flow is geometric (Theorem 3.2), and the same assembly reproduces the independently known Maxwell case (Proposition 3.3); the Type II_∞ crossed product with canonical trace exists and its physical observer-clock reading has the same standing as for conformal matter (Section 4); the diamond's local temperature and IR-safe central phase rate $\Delta E/\hbar$ are helicity-blind (Section 5). The inputs are published theorems [2, 3, 4, 6, 5, 21, 23, 24]; the assembly, the flat-space master-equation anchor, and the consequences are the contribution here. The moral we would draw for the broader program of subregion algebras in gravity is simple: *conformal invariance of the action was never the right hypothesis for geometric diamond modular flow—mode-wise reducibility to theories with known flow is enough, and the physical graviton has it.*

7.2 The general spin-s picture

The assembly uses three inputs, of which only one is field-specific: a decomposition of the gauge-invariant ball content into scalar towers, localized in the ball. This suggests a general statement.

Proposition 7.1 (Conditional spin-s extension)**.** *Let a free massless field on Minkowski space admit a Benedetti–Casini-type reduction: a decomposition of its gauge-invariant ball content into towers of massless-scalar spherical modes (any starting ℓ, any multiplicities), localized in the ball, with the vacuum reducing mode by mode. Then its vacuum diamond modular flow is geometric—the CHM conformal-Killing flow restricted to its towers—by Lemma 3.1 and the per-mode theorem of [5].*

The hypothesis is established for $s = 0$ (trivially), $s = 1$ [6], and $s = 2$ [4], with towers starting at $\ell = s$—the angular-momentum threshold for radiative multipoles of a helicity-s field.

For higher spins and p-forms the corresponding tower structure underlies the hyperbolic-cylinder entropy computations of David and Mukherjee [7], and we expect the reduction to hold with towers at $\ell \geq s$, but a flat-ball derivation at the algebra level (the analogue of statement (R3)) has to our knowledge not been written down for $s \geq 3$; the proposition is stated conditionally for this reason. If the hypothesis holds generally, geometric modular flow in a diamond is the *rule* for free massless field content, not a peculiarity of CFTs—the role of conformal invariance was only ever to make the computation easy.

7.3 Honest residues

1. **Edge modes and the center.** The decomposition (8) is the bulk tower algebra. Gauge theories and gravity assign a sphere additional boundary data—edge modes, or a center of the algebra—which contribute to entropy constants and to the universal logarithmic coefficient, and for the graviton are actively studied [7, 8, 9]. Our flow statement is for the algebra modulo its center; the edge sector does not carry a flow obstruction, but its contribution to the crossed-product trace normalization is open.

2. **The observer is transcribed, not constructed.** For finite diamonds, the bounded-below observer Hamiltonian and clock of [23] are carried over from the de Sitter Killing case by structural analogy. This was the state of the art for conformal matter before our result; our result removes the flow-level graviton-specific obstruction, while graviton-specific subtleties of the transcription itself (constraints, edge sector) remain possible and open.

3. **Trace-anomaly normalization.** The additive normalization of the crossed-product trace (equivalently of the generalized entropy) involves the anomaly coefficients of the field content and is not computed here. For the graviton this interacts with the edge sector; the two open items are likely linked.

4. **No covariant local density.** Equation (12) is a mode sum. A covariant, gauge-invariant local graviton energy density does not exist in the standard sense, and we make no claim to one; applications that would require it (rather than the mode sum) are outside the result.

5. **Linearized order.** Everything here is free-field modular theory. Graviton self-interaction enters at the next order in the gravitational coupling and is not addressed.

6. **For decoherence programs.** The result removes the radiative-sector failure mode of modular-clock decoherence mechanisms (Section 5.3) but does not by itself establish the modular-time identification on which such mechanisms rest; that identification remains a hypothesis at the full operator level [30].

7.4 Outlook

Three directions seem ripe. First, the intrinsic observer: with the flow now known to be geometric for all the field content of a tabletop diamond—scalar, Maxwell, graviton—the remaining work in promoting the transcribed CLPW clock to a construction is the same problem for matter and gravity, and can be attacked once rather than per species. Second, the edge sector: the split between a helicity-blind bulk flow and a field-specific edge/anomaly bookkeeping suggests that the crossed-product trace normalization—and through it the additive entropy constant—may be computable by matching the cylinder partition functions of [7, 8] against the canonical Takesaki trace, a calculation that would also bear on the $O(1)$ coefficient in modular-clock decoherence rates. Third, and most simply, the hypothesis of Proposition 7.1 deserves a direct flat-ball proof for $s \geq 3$: if it holds, the modular geometry of finite regions is universal across the massless spectrum.

8 Conclusion

The modular flow of a causal diamond was known to be geometric for conformally invariant fields and expected to be inaccessible for the one field that gravitational physics cannot do without. We have shown that the expectation rested on mistaking a computational route for a physical requirement. The gauge-invariant content of the linearized graviton in a ball is two towers of massless-scalar spherical modes with $\ell \geq 2$—a published reduction [4] with an elementary anchor in the flat-space degeneration of the Regge–Wheeler and Zerilli equations, verified here symbolically and at machine precision on the lattice. Rotational invariance keeps the vacuum modular operator block-diagonal over the towers, and each tower, though manifestly non-conformal, inherits a local modular Hamiltonian with the conformal-Killing weight $f(r)$ that generates its flow [5]. The graviton's diamond modular flow is therefore geometric, its crossed product is Type II_∞ with a canonical trace, and the modular clock of a finite diamond—including its IR-safe central phase rate $\Delta E/\hbar$—is blind to helicity. The same assembly, run on the Maxwell field where the conformal route is also available, reproduces the known answer: the validation and the generalization come from the same mechanism, and the helicity ladder two scalars $\supset$ Maxwell $\supset$ graviton is exact mode by mode.

What remains open is stated plainly: the edge-mode/center sector and the trace-anomaly normalization of the crossed-product trace; the intrinsic (rather than transcribed) construction of the diamond observer; the flat-ball reduction hypothesis for spins $s \geq 3$; and everything beyond linearized order. None of these is a graviton-specific obstruction at the level of the flow any longer. That, we think, is the structural point: the modular theory of finite regions, and the Type II algebras built from it, extend to the physical gravitational field, and the problems that remain are the problems of the subject as a whole.

Appendices

A Master potentials and the flat-space degeneration

For completeness we record the master equations and the degeneration used in Section 3.2. Perturbations of a Schwarzschild background of mass M (units $G = c = 1$) decompose into odd- and even-parity sectors per (ℓ, m); each sector reduces to a single master field $\psi(t, r)$ obeying

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial r_*^2} + V(r)\,\psi = 0, \qquad \frac{dr_*}{dr} = \Big(1 - \frac{2M}{r}\Big)^{-1}, \tag{18}$$

with $V = V_{\mathrm{RW}}$ (odd parity, Regge–Wheeler [10]), $V = V_{\mathrm{Z}}$ (even parity, Zerilli [11]), as given in Eqs. (9)–(10), while a minimally coupled massless scalar has $V = V_{\mathrm{s}} = (1 - 2M/r)[\ell(\ell+1)/r^2 + 2M/r^3]$.

Degeneration at $M = 0$. Setting $M = 0$: the tortoise coordinate becomes $r_* = r$; $V_{\mathrm{RW}} \to \ell(\ell+1)/r^2$ and $V_{\mathrm{s}} \to \ell(\ell+1)/r^2$ by inspection; and for Zerilli,

$$V_{\mathrm{Z}}\big|_{M=0} = \frac{2\lambda^2(\lambda+1)r^3}{r^3\,\lambda^2 r^2} = \frac{2(\lambda+1)}{r^2} = \frac{(\ell-1)(\ell+2)+2}{r^2} = \frac{\ell(\ell+1)}{r^2}, \tag{19}$$

using $\lambda = (\ell-1)(\ell+2)/2$. The common flat master equation $-\psi'' + [\ell(\ell+1)/r^2]\psi = \omega^2\psi$ is the Riccati–Bessel equation, solved by $\psi = r\,j_\ell(\omega r)$, regular at the origin ($\psi \sim r^{\ell+1}$)—precisely the radial mode functions of the dimensionally reduced massless scalar (6).

Distinctness at $M \neq 0$. At $M \neq 0$ the three potentials are pairwise distinct; V_{RW} and V_{Z} are isospectral via a Darboux transformation [13, 14] but not equal, and neither equals V_{s}. The flat-space identity of the graviton parities with the scalar is thus a genuine degeneration at $M = 0$, not an artifact of a shared definition—which is what makes its lattice verification (bitwise equality of couplings built from the three *different* formulas at $M = 0$, against $O(10^{-2})$ differences at $M = 0.1$) a meaningful check.

Relation to the Benedetti–Casini variables. The master fields above arise from the Regge–Wheeler gauge-fixing of $h_{\mu\nu}$ on Schwarzschild (tensor-harmonic conventions as in [10, 12]); the variables of [4] arise from a gauge fixing adapted to the $t = 0$ ball in flat space. At $M = 0$ both procedures land on the same reduced dynamics—two decoupled scalar towers with $\ell \geq 2$—as they must, since the gauge-invariant content per (ℓ, m) is two radiative modes and the reduced dynamics is gauge-independent. We use the master-equation route as the anchor because its $M \to 0$ limit makes the equivalence elementary and machine-checkable.

B Lattice method and check inventory

Discretizations. Each reduced mode is a half-line theory $H = \frac{1}{2}\int dr\,[\pi^2 + (\partial_r\psi)^2 + V(r)\psi^2]$, discretized on N sites at unit spacing as $H = \frac{1}{2}\pi^{\mathsf{T}}\pi + \frac{1}{2}\psi^{\mathsf{T}}K\psi$. We use two schemes. (i) *Srednicki's* [25]: $K_{11} = \frac{9}{4} + \ell(\ell+1)$, $K_{jj} = [(j-\frac{1}{2})^2 + (j+\frac{1}{2})^2 + \ell(\ell+1)]/j^2$, $K_{j,j+1} = -(j+\frac{1}{2})^2/[j(j+1)]$, the standard benchmark for the area law. (ii) A direct *V-scheme* for the master fields: $K_{jj} = 2 + V(j)$, $K_{j,j+1} = -1$, Dirichlet at both ends, with V taken from the mode's own potential formula—V_{s}, $V_{\mathrm{RW}}|_{M\to 0}$, or $V_{\mathrm{Z}}|_{M\to 0}$—so that scalar and graviton couplings are computed through different expressions and their equality is informative.

Reduced state and entanglement data. Ground-state correlators are $X = \frac{1}{2}K^{-1/2}$, $P = \frac{1}{2}K^{1/2}$. Restricting to the inner $n < N$ sites $(X_A,\ P_A)$, the symplectic eigenvalues $\nu_k \geq \frac{1}{2}$ are the positive square roots of the eigenvalues of $X_A^{1/2} P_A X_A^{1/2}$, and the mode entropy is $S = \sum_k [(\nu_k + \frac{1}{2}) \ln(\nu_k + \frac{1}{2}) - (\nu_k - \frac{1}{2}) \ln(\nu_k - \frac{1}{2})]$ [26]. The modular operator of the reduced quasi-free state is determined by the pair (X_A, P_A) together with the symplectic form [17, 18], which is what makes the per-mode equality of (X_A, P_A) an equality of modular data.

Check inventory. The numerical suite comprises 42 checks in five blocks: (A) scalar baseline—positivity of K_ℓ, the Heisenberg bound $\nu_k \geq \frac{1}{2}$, the area-law coefficient $S/(n+\frac{1}{2})^2 \in [0.290, 0.300]$ at $n = 10, 20$ ($N = 60$, summed to angular-momentum convergence), monotonicity of S_ℓ in ℓ; (B) per-mode identity—bitwise equality of K from the three potential formulas at $M = 0$ for $\ell = 2, \ldots, 6$ and both parities, trace invariants $\mathrm{Tr}[(X_A P_A)^k]$, $k \leq 4$, at 10^{-15}, symplectic spectra and entropies at 10^{-12}, and pairwise distinctness at $M = 0.1$; (C) mode bookkeeping—the tower count $2[(L+1)^2 - 4]$, the ADM count, the missing-modes count; (D) tower sums—the finite-sum identity (17) at 10^{-12} with independently computed sides, positivity of the $\ell = 0, 1$ deficit, and stability across $N = 40, 60, 100$; (E) the helicity ladder—Maxwell mode counts $2[(L+1)^2 - 1]$, the ladder deficits (2 modes scalars → Maxwell, 6 modes Maxwell → graviton), and the entropy-ladder identities $S_{2\,\mathrm{scalars}} - S_{\mathrm{Maxwell}} = 2S_0$, $S_{\mathrm{Maxwell}} - S_{\mathrm{grav}} = 6S_1$ at 10^{-12}, with strict ordering. The symbolic companion verifies the degeneration (11) at general symbolic ℓ with identically vanishing residuals (ten identities).

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